Consistent Aggregation With Superlative and Other Price Indices
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1 Motivation and Background

- Often we want to decompose the overall inflation into sector specific inflation rates.
- For example, central banks decompose the overall inflation into the
  - core inflation (all products except energy and seasonal food) and the
  - non-core inflation (seasonal food and energy).
- A price index should give the same result with and without decomposition.
- Then the price index is said to be consistent in aggregation.
A very restrictive notion of consistency in aggregation has been introduced by Vartia (1976a, b).

Blackorby and Primont (1980) develop a far less restrictive version.

Auer (2004) proposes a compromise between Vartia and Blackorby/Primont.

2 Basic Principle of Two Stage Aggregation

- Set of items: \( S = (1, \ldots, N) \).
- Laspeyres index:

\[
P^{La} = \sum_{i \in S} r_i \frac{v_i^0}{\sum_{j \in S} v_j^0}
\]

with \( r_i = p_i^1 / p_i^0 \) and \( v_i^0 = p_i^0 q_i^0 \).
- When \( N = 1 \), then all sensible price indices give \( P = r_1 \).
- Therefore, \( r_i \) is denoted as the primary attribute of the price index (Blackorby and Primont, 1980).
- The other attributes of a price index are secondary attributes (denoted by \( z_i^1, z_i^2, \ldots \))
- The Laspeyres index has only one secondary attribute \( z_i^1 = v_i^0 \).
## 2. Basic Principle of Two Stage Aggregation

<table>
<thead>
<tr>
<th>Single stage compilation</th>
<th>Two stage compilation</th>
</tr>
</thead>
<tbody>
<tr>
<td>((r_1, \ldots, r_N), (v_1^0, \ldots, v_N^0) , (v_1^1, \ldots, v_N^1))</td>
<td>((r_1, \ldots, r_N), (z_1^1, \ldots, z_N^1))</td>
</tr>
<tr>
<td>(P_{La} = \sum_{i \in S} r_i \frac{v_i^0}{\sum_{j \in S} v_j})</td>
<td>(P_{La} = \sum_{i \in S} r_i \frac{z_i^1}{\sum_{j \in S} z_j^1})</td>
</tr>
<tr>
<td>(z_i^1 = v_i^0)</td>
<td>(P_{k} = \sum_{i \in S_k} r_i \frac{z_i^1}{\sum_{j \in S_k} z_j^1})</td>
</tr>
<tr>
<td>(Z_1^1 = \sum_{i \in S_k} z_i^1)</td>
<td>(Z_k^1 = \sum_{i \in S_k} z_i^1)</td>
</tr>
<tr>
<td>(P_{La} = \sum_{k=1}^K P_k \frac{Z_k^1}{\sum_{l=1}^K Z_l^1})</td>
<td>(P_{La} = \sum_{k=1}^K P_k \frac{Z_k^1}{\sum_{l=1}^K Z_l^1})</td>
</tr>
</tbody>
</table>
3 Illustrative Example

- Swedish CPI Data from the base period 2010 \((t = 0)\) and the comparison period 2011 \((t = 1)\).
- \(S = 1, 2, \ldots, 360\) items (four-digit level COICOP classification)
- \(S_1 = 1, 2, \ldots, 301\) are the items assigned to core inflation.
- \(S_2 = 302, \ldots, 360\) are the items assigned to non-core inflation.
- For each item we know \((r_i, \nu^0_i, \nu^1_i)\).
## Table 1: Two Stage Aggregation of Laspeyres Index

<table>
<thead>
<tr>
<th></th>
<th>BASIC HEADING INFORMATION</th>
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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>i</td>
<td>COICOP</td>
<td>GROUP</td>
<td>PRODUCT</td>
<td>( r_i )</td>
</tr>
<tr>
<td>( S_1 ) (CORE)</td>
<td>1</td>
<td>01.1.1</td>
<td>1113</td>
<td>Wheat Bread</td>
<td>1.0333</td>
</tr>
<tr>
<td></td>
<td>301</td>
<td>12.7</td>
<td>9704</td>
<td>Lawyer Fees</td>
<td>1.0282</td>
</tr>
</tbody>
</table>

\[ P_1^{La} = 1.0268 \]

\[ Z_1^1 = 1243742 \]

<p>| | | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_2 ) (OTHER)</td>
<td>302</td>
<td>01.1.3</td>
<td>1307</td>
<td>Herring</td>
<td>1.0438</td>
<td>155</td>
<td>128</td>
<td></td>
<td>155</td>
<td></td>
</tr>
<tr>
<td></td>
<td>360</td>
<td>07.2.2</td>
<td>6225</td>
<td>E 85 Fuel</td>
<td>1.0479</td>
<td>1205</td>
<td>1245</td>
<td></td>
<td>1205</td>
<td></td>
</tr>
</tbody>
</table>

\[ P_2^{La} = 1.0366 \]

\[ Z_2^1 = 182775 \]

3. Illustrative Example

- Single stage aggregation by the Laspeyres index:

\[ P^{La} = \sum_{i \in S} r_i \frac{z^1_i}{\sum_{j \in S} z^1_j} = 1.028025 \]

with \( z^1_i = v_i^0 \).

- Second stage of two stage aggregation by the Laspeyres index:

\[ P^{La} = \sum_{k=1,2} P^{La}_k \frac{Z^1_k}{\sum_{l=1,2} Z^1_l} = 1.028025 \]

- Laspeyres index is consistent in aggregation with respect to the secondary attribute \( z^1_i = v_i^0 \).
4 Superlative Price Indices

- Fisher index

\[ P^{Fi} = \left( \frac{\sum_{i \in S} v_i^0 r_i}{\sum_{i \in S} v_i^0} \cdot \frac{\sum_{i \in S} v_i^1}{\sum_{i \in S} v_i^1 / r_i} \right)^{1/2} \]

- Törnqvist index

\[ \ln P^{Tö} = \sum_{i \in S} \ln (r_i) \cdot \frac{1}{2} \left( \frac{v_i^0}{\sum_{j \in S} v_j^0} + \frac{v_i^1}{\sum_{j \in S} v_j^1} \right) \]

- Walsh index

\[ P^{Wa} = \sum_{i \in S} r_i \sqrt{v_i^0 v_i^1 / r_i} \]
4. Superlative Price Indices

\[ P^s_i = \left( \frac{\sum_{i \in S} v_i^0 r_i}{\sum_{i \in S} v_i^0} \frac{\sum_{i \in S} v_i^1 / r_i}{\sum_{i \in S} v_i^1} \right)^{1/2} \]

\[ z_i^1 = v_i^0, \quad z_i^2 = v_i^1, \quad z_i^3 = v_i^0 r_i, \quad z_i^4 = v_i^1 / r_i \]

**single stage**

\[ P^s_i = \left( \frac{\sum_{i \in S} z_i^3}{\sum_{i \in S} z_i^1} \frac{\sum_{i \in S} z_i^2}{\sum_{i \in S} z_i^4} \right)^{1/2} \]

**two stage**

\[ P^s_{k_i} = \left( \frac{\sum_{i \in S_k} z_i^3}{\sum_{i \in S_k} z_i^1} \frac{\sum_{i \in S_k} z_i^2}{\sum_{i \in S_k} z_i^4} \right)^{1/2} \]

\[ Z_k^1 = \sum_{i \in S_k} z_i^1, \quad \ldots, \quad Z_k^4 = \sum_{i \in S_k} z_i^4 \]

\[ P^s_i = \left( \frac{\sum_{k=1}^{K} Z_k^3}{\sum_{k=1}^{K} Z_k^1} \frac{\sum_{k=1}^{K} Z_k^2}{\sum_{k=1}^{K} Z_k^4} \right)^{1/2} \]
Fisher index is consistent in aggregation with respect to
\( z_i^1 = v_i^0 \), \( z_i^2 = v_i^1 \), \( z_i^3 = v_i^0 r_i \), and \( z_i^4 = v_i^1 / r_i \).

Possible objections:

- primary attribute is missing in index formula
- \( z_i^3 = z_i^1 r_i \), and \( z_i^4 = z_i^2 / r_i \), but
  \( Z_k^3 \neq Z_k^1 P_k \), and \( Z_k^4 \neq Z_k^2 / P_k \).
- secondary attributes must be either \( v_i^0 \) or \( v_i^1 \).
### 4. Superlative Price Indices

#### Single Stage

\[ P_{Wa} = \sum_{i \in S} r_i \frac{\sqrt{v_i^0 v_i^1}}{r_i} \]

\[ z_i^1 = \frac{\sqrt{v_i^0 v_i^1}}{r_i} \]

#### Two Stage

\[ P_{Wa} = \sum_{i \in S_k} r_i \frac{z_i}{\sum_{j \in S_k} z_j} \]

\[ Z_k^1 = \sum_{i \in S_k} z_i^1 \]

\[ Z_k^1 = \sum_{i \in S_k} z_i \]

\[ P_{Wa} = \sum_{k=1}^{K} P_k \frac{Z_k}{\sum_{l=1}^{K} Z_l} \]
Walsh index is consistent in aggregation with respect to
\[ z_i^1 = \sqrt{v_i^0 v_i^1 / r_i}. \]
Possible objections:
- secondary attributes must be either \( v_i^0 \) or \( v_i^1 \).
## 5 Other Price Indices

### Table 2: More Price Indices That Are Consistent in Aggregation

<table>
<thead>
<tr>
<th>Name</th>
<th>Price Index Formula</th>
<th>Function</th>
<th>Secondary Attributes</th>
<th>Attributes $z_i^q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laspeyres</td>
<td>$P^{La} = \frac{\sum v^0_i r_i}{\sum v^0_i}$</td>
<td>$r_i z_i$</td>
<td>$v^0_i$</td>
<td></td>
</tr>
<tr>
<td>Paasche</td>
<td>$P^{Pa} = \frac{\sum v^1_i}{\sum v^1_i / r_i}$</td>
<td>$r_i^{-1} z_i$</td>
<td>$v^1_i$</td>
<td></td>
</tr>
<tr>
<td>Marshall-</td>
<td>$P^{ME} = \sum r_i \frac{v^0_i + v^1_i / r_i}{\sum (v^0_i + v^1_i / r_i)}$</td>
<td>$r_i z_i$</td>
<td>$(v^0_i + v^1_i / r_i)$</td>
<td></td>
</tr>
<tr>
<td>Edgeworth</td>
<td>$\ln P^{Wa2} = \sum \ln r_i \frac{\sqrt{v^0_i v^1_i}}{\sum \sqrt{v^0_i v^1_i}}$</td>
<td>$(\ln r_i) z_i$</td>
<td>$\sqrt{v^0_i v^1_i}$</td>
<td></td>
</tr>
</tbody>
</table>
Table 2: (contin.)

\[
\begin{align*}
\text{Walsh-Vartia} & : \ln P^{WV} = \sum \ln r_i \frac{\sqrt{v_i^0}}{\sqrt{\sum v_j^0}} \frac{\sqrt{v_i^1}}{\sqrt{\sum v_j^1}} \quad (\ln r_i) \sqrt{z_i^1 z_i^2} \quad v_i^0, v_i^1 \\
\text{Theil} & : \ln P^{Th} = \sum \ln r_i \frac{\sqrt[3]{\frac{1}{2}(v_i^0 + v_i^1)v_i^0 v_i^1}}{\sqrt[3]{\frac{1}{2}(v_j^0 + v_j^1)v_j^0 v_j^1}} \quad (\ln r_i) z_i \quad 3^{\frac{1}{2}} \sqrt[3]{\frac{1}{2}(v_i^0 + v_i^1)v_i^0 v_i^1} \\
\text{Vartia} & : \ln P^{Va} = \sum \ln r_i \frac{L(v_i^0, v_i^1)}{L(\sum v_i^0, \sum v_i^1)} \quad (\ln r_i) L(z_i^1, z_i^2) \quad v_i^0, v_i^1 \\
\text{with}^* & : \quad L(a, b) = \begin{cases} 
\frac{b - a}{\ln b - \ln a} & \text{for } a \neq b \\
\frac{a}{\ln a} & \text{for } a = b
\end{cases}
\end{align*}
\]
Table 3: Generalized Unit Value (GUV) Indices

<table>
<thead>
<tr>
<th>Name</th>
<th>Price Index Formula</th>
<th>Function $f(r_i, z_i^1, ..., z_i^Q)$</th>
<th>Secondary Attributes $z_i^q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Banerjee</td>
<td>$P_{Ba} = \frac{\sum v_i^1 \cdot \sum v_i^0 (1 + r_i)}{\sum v_i^0 \cdot \sum v_i^1 (1 + 1/r_i)}$</td>
<td>$r_i \frac{z_i^1}{z_i^2} z_i^3$</td>
<td>$v_i^0, v_i^1, v_i^1 \frac{1 + r_i}{r_i}$</td>
</tr>
<tr>
<td>(GUV-3)**</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Davies</td>
<td>$P_{Da} = \frac{\sum v_i^1 \cdot \sum v_i^0 \sqrt{r_i}}{\sum v_i^0 \cdot \sum v_i^1 \sqrt{1/r_i}}$</td>
<td>$r_i \frac{z_i^1}{z_i^2} z_i^3$</td>
<td>$v_i^0, v_i^1, v_i^1 / \sqrt{r_i}$</td>
</tr>
<tr>
<td>(GUV-4)**</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(GUV-5)**</td>
<td>$P_{GUV-5} = \frac{\sum v_i^1 \cdot \sum v_i^0 (1 + r_i^{-1})^{-1}}{\sum v_i^0 \cdot \sum v_i^1 (1 + r_i)^{-1}}$</td>
<td>$r_i \frac{z_i^1}{z_i^2} z_i^3$</td>
<td>$v_i^0, v_i^1, v_i^1 / (r_i + 1)$</td>
</tr>
<tr>
<td>(GUV-5)**</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 3: (contin.)

\begin{align*}
(P_{GUV-6})^{**} & \quad P_{GUV-6}^{**} = \frac{\sum v_{i}^{1} \sum v_{i}^{0} r_{i}^1/(v_{i}^{0}+v_{i}^{1})}{\sum v_{i}^{0} \sum v_{i}^{1} r_{i}^1/v_{i}^{1}/(v_{i}^{0}+v_{i}^{1})} \\
Lehr & \quad P_{Le} = \frac{\sum v_{i}^{1} \sum v_{i}^{0} (v_{i}^{0} + v_{i}^{1})(v_{i}^{0} + v_{i}^{1}/r_{i})}{\sum v_{i}^{0} \sum v_{i}^{1} (v_{i}^{0} + v_{i}^{1})(v_{i}^{0} r_{i} + v_{i}^{1})}
\end{align*}

\begin{align*}
&v_{i}^{0}, v_{i}^{1}, r_{i}^1, v_{i}^{0}/v_{i}^{1} \\
&v_{i}^{0}, v_{i}^{1}, v_{i}^{0} r_{i}/v_{i}^{0} + v_{i}^{1}
\end{align*}
6 Additional Requirements

- In contrast to Blackorby and Primont (1980), we allow only for secondary attributes that are functions of no other information than $r_i$, $v_i^0$ and $v_i^1$.

- **Requirement A:** Secondary attributes should represent monetary values (e.g., $v_i^0$ or $\sqrt{v_i^0 v_i^1}$, but not $v_i^0 v_i^1$).

- **Requirement B:** The secondary attributes are aggregated additively: $Z_k^q = \sum_{i \in S_k} z_i^q$.

- **Requirement C:** Any functional relationship between the secondary attributes of the individual items must carry over to the aggregated secondary attributes.

- This eliminates the indices of Table 3, the Fisher index, but not the Walsh index.

- The Walsh index is “ABC-consistent in aggregation”.
Requirement D: (Auer, 2004) Only the secondary attributes \( v_i^0, v_i^1, v_i^0 r_i \) and \( v_i^1 / r_i \) are admissible (note that \( v_i^0 r_i = p_i^1 q_i^0 \) and \( v_i^1 / r_i = p_i^0 q_i^1 \)).

This eliminates the Walsh, the Walsh-2, and the Theil index.

Requirement E: (Vartia, 1976a,b, Balk 1995, Pursiainen 2005, 2008) Only the secondary attributes \( v_i^0 \) and \( v_i^1 \) are admissible.

This eliminates the Marshall-Edgeworth index.

Then we are left with the Laspeyres, Paasche, Walsh-Vartia, and Vartia index.
7 Concluding Remarks

- Very heterogeneous definitions of consistency in aggregation have been proposed in the literature.
- We have introduced a rigorous formalization of this notion that allows to compare these definitions.
- Our definition of consistency in aggregation can be made more restrictive by attaching additional requirements.
- The Walsh index satisfies the three least controversial of these requirements.