How Much Does Formula vs. Chaining Matter for a Cost-of-Living Index? The CPI-U vs. the C-CPI-U*

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Abstract

A large economics literature has debated the best formula to estimate a cost-of-living index (COLI) - this study shows that formula may not be relevant for many purposes for an index chained at a monthly frequency if current weight information is properly used. The large majority of the difference between the levels of the CPI-U and the generally lower C-CPI-U (a COLI) is due to the CPI-U weights holding quantities constant over long periods, rather than the difference in formula assumptions. A new method to avoid chain drift with long term price relatives is developed to effectively approximate a COLI.

JEL Classifications: C43, C82, E31

Keywords: Consumer Price Index, CPI, Divisia, Index Number, Inflation, Price Index, Cost of Living

1 Introduction

This study explains the difference between the CPI-U and the C-CPI-U (the 'chained' CPI-U). The C-CPI-U is typically and cumulatively lower than the CPI-U. While both indexes use the same elementary item-area indexes as inputs, they use different aggregation formulas and different weights to estimate aggregate price change.

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It is demonstrated that the large majority of the difference between the CPI-U and the C-CPI-U levels is due to the weighting differences, rather than the aggregation formula difference. In particular, weights in the CPI-U implicitly constrain quantities to be constant over long periods of time, while weights in the C-CPI-U allow quantities to change every month. Differences over only one month period at a time, whether it be holding a quantity or share weight constant, or a geometric vs. arithmetic mean, matter very little since prices change very little over one month. But the CPI-U weights hold quantities constant for 36 months on average, so the total effect of this dominates since it is roughly on the order of 36 times of a one month difference.

A major question in the economics literature, including Pollak (1971), Samuelson and Swamy (1974), Diewert (1976), Diewert (1978), and as discussed in the CPI Manual (ILO 2004), among many others, is the appropriate formula to use in measuring cost-of-living-indexes (COLIs). But the results here indicate that for at least CPI aggregation, the formula for approximating a COLI isn't very important to get effectively the same results; all that is necessary is that aggregation weights are updated such that quantities are not held constant over long periods of time, and long term price relatives are used where needed.

Section 2 describes the CPI-U and C-CPI-U, the differences between them, and the effects of formula differences and chaining with current weights. Section 3 gives a brief overview of the methods used to breakdown the differences by making intermediate indexes which change by one difference at a time, and an overview of results. Section 4 describes in detail the methods of changing the weights in intermediate indexes before formula, and section 5 describes the methods of changing the formula first. Section 6 concludes.

2 The CPI-U and C-CPI-U: COGI versus COLI

At the upper level of aggregation the CPI-U uses a Lowe, or 'modified Laspeyres' formula, which is a Laspeyres index with lagged weights. The Lowe formula is a cost-of-goods index, or COGI: as described in "At What Price?: Conceptualizing and Measuring Cost-of-Living and Price Indexes", pp. 2-3. A COGI is the ratio of the expenditure needed in the current period to the expenditure needed in the past period to buy a fixed basket of goods. The CPI-U uses an implicit basket of goods defined over a multi-year period base period, which is updated every several years with a lag for processing. The CPI-U index relative between months t-1 and t is

$$CPIU_{t-1,t} = \frac{\Sigma_{ia} Aggweight_{iaB} I_{iat}}{\Sigma_{ia} Aggweight_{iaB} I_{i,a,t-1}} \quad (1)$$

where I_{iat} denotes the item-area cell index level for item i in area a for month t, and $Aggweight_{iaB}$ denotes the aggregation weight for item i in area a for base period B. The aggregation weight is total expenditure over the base period, measured by the Consumer Expenditure Survey (CE), divided by the average index level for that item-area over the base period.

The Lowe formula can be rewritten as an arithmetic mean of item-area index relatives, weighted by expenditure shares that hold the implicit quantity constant,

$$CPIU_{t-1,t} = \Sigma_{ia} \frac{Aggweight_{iaB} I_{i,a,t-1}}{\Sigma_{ia} Aggweight_{iaB} I_{i,a,t-1}} R_{iat} \tag{2}$$

where $\frac{Aggweight_{iaB} I_{i,a,t-1}}{\Sigma_{ia} Aggweight_{iaB} I_{i,a,t-1}}$ is an expenditure share updated from the base period B by the index levels $I_{i,a,t-1}$. Therefore, the Lowe weights are updated shares, but those updates do not use new expenditure information – they only use the index relatives. The shares are only updated in such a way that quantities are not updated.

Conversely, at the upper level of aggregation, the C-CPI-U uses the Tornqvist formula. The Tornqvist formula is meant to be an approximation to a cost of living index, or COLI.[1] As defined in Konus (1939) and Pollak (1983), a COLI is the ratio of the expenditure needed with current prices to the expenditure needed in the past period to purchase a base standard of living, meaning that a consumer would be indifferent to choosing between these two expenditure-price combinations.[2] As described in Diewert (1976), the Tornqvist approximates an arbitrary COLI by approximating the function defining the consumer's necessary expenditure (or cost) given prices and a standard of living.[3]

The Tornqvist formula is

$$T_{t-1,t} = \Pi_i \left(\frac{p_{it}}{p_{i,t-1}} \right)^{\frac{s_{it}+s_{i,t-1}}{2}} \tag{3}$$

, where p_{it} denotes the price of item i in period t, and s_{it} denotes the expenditure share of item i in period t.

Using item-area cells in place of items and item-area index relatives in place of price relatives, and denoting the price relative for item i in area a

[1] The C-CPI-U is actually a Conditional-Cost-of-Living index, because many things that affect the standard of living, including weather, crime, various government services, etc., as described in "At What Price?: Conceptualizing and Measuring Cost-of-Living and Price Indexes" pp. 94-105, are out of the scope of the index. The CPI-U and all other indexes constructed herein are also conditional indexes, and have the same scope.

[2] Balk and Diewert (2003) show that a Lowe index can be considered an approximation of a COLI, but only if there are no significant relative price trends in the data. Below it will be shown that these different trends in the data do make a difference.

[3] From Diewert (1976), Theorem (2.16), p. 122, this approximation represents the expenditure as a second order translog function,

$$\ln C\left(u;p\right) \equiv \alpha_0^* + \Sigma_{i=1}^N \alpha_i^* \ln p_i + \frac{1}{2} \Sigma_{j=1}^N \Sigma_{k=1}^N \gamma_{jk}^* \ln p_j \ln p_k + \beta^* \ln u + \delta^* \left(\ln u\right)^2 + \Sigma_{i=1}^N \varepsilon_i^* \ln u \ln p_i$$

Where i, j, and k denote items, with N total items, u denotes utility, p_i denotes the price of item i, and the $\alpha, \beta, \gamma, \varepsilon$ terms are parameters. This implies the Tornqvist formula, evaluated at $u = \sqrt{u_{t-1} u_t}$.

There are other approximations to COLIs, but the indexes in the class of the Tornqvist, superlatives, are generally found to be close to each other. See Diewert (1978).

between months $t-1$ and t as R_{iat} and the CE measured expenditure share for item i in area a for month t as s_{iat}, this implies the formula for the C-CPI-U index relative of

$$T_{t-1,t} = \Pi_{ia} R_{iat}^{\frac{s_{iat}+s_{i,a,t-1}}{2}} \tag{4}$$

.

The CPI-U and C-CPI-U use the same item-area index relatives for price relatives, and both use CE survey measured expenditures to construct the weights.[4] Thus, the two indexes differ in two ways: weights and formula. The Lowe index is an arithmetic mean formula of index relatives with weights updated by holding the base period quantity fixed, while the Tornqvist is a geometric mean formula of index relatives with a direct measure of the average of current and previous period share weights.

There are three types of differences in the weights used in the CPI-U versus the C-CPI-U: time span of data, lag, and frequency of updating. The CPI-U weights are derived from a multiple year base period. The total expenditure for each item-area over that period is used to derive an aggregation weight that is used in the index. The quantities in those shares are fixed by the base period expenditure, updated by the inflation of the item-area index relatives. For 1998 through 2001, there was a three year base period, and since 2002 it has used a series of two-year base periods. Conversely, the C-CPI-U weights are derived from only two months of data, the current and previous months. Using current shares instead of the share updating of the Lowe index allows the implicit quantity weights to change, even if the new shares don't change. For example, if the current period shares don't differ from the older ones, but the index relative has risen by 2%, then the implicit quantity weight has fallen by 2%.

The second is that the CPI-U weights are updated with a long lag, due chiefly to the processing time needed. For 1998, the lag was two years after the end of the base period, and since 2002, it was one year since the end of the base period. Therefore, the total lag for a given month has been two to five years from 1999 through 2001 and then one to three years since 2002. The C-CPI-U weights, however, have no lag, since they use information from the current (and previous) month. Of course, the C-CPI-U cannot be computed in real time. Third, before 1998, the CPI-U weights were updated around once every 10 years. The weights were updated once in 1998, and then every two years starting in 2002, while the C-CPI-U weights are updated every month.

To the extent that the Tornqvist formula approximates a COLI, those differences that make a COGI different from a COLI would explain the differences between the CPI-U and the C-CPI-U. A COLI index with a base at the

[4] Due to the time needed to process the CE, the final C-CPI-U is made with a two year lag. Only initial and interim estimates of what the final index will be are published earlier.

It has been argued that the C-CPI-U is not a true Tornqvist due to the fact that the variance in the CE survey measured weights (which is due to a low sample size), is smoothed before use in the C-CPI-U. However, it has been found to make almost no difference when the raw weights are actually used.

previous period, for preferences that are constant over that period, will be as low or lower than a COGI over the same period which has a base basket in the previous period, due to consumer substitution. This is because if there are relative price changes, consumers can substitute relatively cheaper goods for relatively more expensive ones, and be better off than if they purchased a fixed basket. Therefore, the cost of maintaining the same standard of living is as low or lower than the cost of buying the same basket of goods.

This was formalized by Konus (1939) who showed that for preferences that don't change over the period, a Laspeyres index with a base in the previous period is an upper bound for a COLI with the same base period, since a Laspeyres index is a COGI with the basket fixed in the previous period. He also showed that a Paasche index, which is a COGI with the base basket in the current period, is a lower bound for a COLI with a base in the current period, also because of consumer substitution. The Laspeyres and Paasche both bound a COLI with a base in-between the two periods.

The Tornqvist formula has two ways to incorporate changes in consumer purchases to approximate a COLI: (i) the current share weights are direct information on changes in consumer purchases that do not hold quantities constant over long periods of time, while the quantity weights in the Lowe formula are only information on past purchases that do not change; (ii) the use of a geometric mean to aggregate the item-area indexes instead of the Lowe/Laspeyres arithmetic mean assumes a certain pattern of consumer substitution. Because different weights could be used in either an arithmetic or geometric mean, for clarity and simplicity (i) will be referred to as the effects of weights, while (ii), use of an arithmetic vs. geometric mean, will refer to the effects of formula.

Updating quantity weights more frequently will make a Lowe index fall, as described in Greenlees & Williams (2009). Since consumers substitute away from items with rising prices, those items with higher inflation will have relatively falling quantities in the long run, all else equal. Therefore updated weights in a Lowe index will give lower weight to higher inflation goods, and lower the long run index because of consumer substitution.

This fall in a Lowe index towards a COLI is not a coincidence. The CPI-U is effectively chained biennially, and the C-CPI-U is chained at the monthly frequency. However, a Lowe index with more frequently updated weights will approach a monthly chained Laspeyres.[5] As shown by Diewert (1978), Diewert (1980), and Balk (2005), both the Laspeyres and the Tornqvist formulas (as well as the Paasche and others), approximate a Divisia index, introduced by Divisia (1925),[6] which is a price index for continuous time,

$$P_{t',t}^{Div} = \exp\left(\int_{t'}^{t} \Sigma_{i=1}^{N} s_i(\tau) \frac{d\ln p_i(\tau)}{d\tau} d\tau\right) \tag{5}$$

where $s_i(\tau)$ denotes the share of item i at point in time τ, and $p_i(\tau)$ denotes

[5] Balk and Diewert (2003) discuss the relation ship between the Lowe and Laspeyres indexes.

[6] Divisia (1926) also showed that the Laspeyres index was a first order approximation to his index.

the price of item i at τ. Because all of the discrete time indexes approximate a Divisia index around a point where all the prices have no change, as the time interval for a Laspeyres or Tornqvist index shrinks, they both approach the Divisia index and thus approach each other. The above studies point out that in that sense, as the interval shrinks, the formula doesn't matter.

At a given point in time, the change in the Divisia index is in fact the "instantaneous" cost-of-living index at that point. Denoting the expenditure needed at a price vector $p(\tau)$ at time τ to obtain the standard of living or utility level $u(\tau)$ as $e(p(\tau), u(\tau))$, the expenditure function at $p(\tau)$ and $u(\tau)$, note that the integrand of (5),

$$\Sigma_{i=1}^{N} s_i(\tau) \frac{d \ln p_i(\tau)}{d\tau} = \frac{\partial \ln e(p(\tau), u(\tau))}{\partial p(\tau)} \tag{6}$$

, where $\frac{\partial \ln e(p(\tau),u(\tau))}{\partial p(\tau)}$ can be considered as an "instantaneous" analogy to a COLI $\frac{e(p(\tau'),u(\tau))}{e(p(\tau),u(\tau))}$ between time periods τ and τ'.[7] As described in Diewert (1983), since the COLI with a base reference period is bounded above by the Laspeyres, and a COLI with a current reference is bounded below by the Paasche, as the interval shrinks, both indexes approach the COLI at that point. Since the Tornqvist approximates a COLI at the point where prices are unchanged, and Balk and Diewert (2003) also show that both the Laspeyres (as a specific case of the Lowe index) approximates a COLI at that same point, both indexes approach the instantaneous COLI as the interval shrinks. If official price indexes are supposed to report the current inflation rate, then as the interval shrinks they approach the appropriate target. The only question is how close to the target does monthly chaining get.

The goal of this project is to determine how important quantity weight updating and the geometric vs. arithmetic mean formula are in moving the CPI-U to the C-CPI-U. This analysis implies that the formula plays a minor role in causing the divergence between the CPI-U and the C-CPI-U. Instead, it is using expenditure share weights that don't hold quantities constant over long periods that are responsible for the majority of the divergence. The qualitative nature of this result holds across three different approaches to breaking down the divergence.

[7] The instantaneous COLI at a point in time should not be confused with a direct COLI over a discrete time interval. As discussed by Samuelson and Swamy (1974) and Diewert (2004), the Divisia index (which is an integral over time) will only equal a direct COLI over that time period if consumer preferences are homothetic. Thus, a chained (or integrated over continuous time) index such as the Lowe or Tornqvist, even as the interval shrinks, will only be the true COLI over that period in the case that preferences are homothetic. However, Reinsdorf (1998b) shows that the Divisia index is relevant in a more general sense even if preferences are not homothetic.

3 Breakdown Overview

To explain how much the weight and formula effects matter for the different levels of the indexes, a number of intermediate indexes were constructed which each incorporated a different change in the CPI-U which either made no significant difference or made it more like the C-CPI-U. The total difference between the intermediate indexes relative to the total difference between the CPI-U and C-CPI-U is then considered the effect of that change.

There is more than one way to move from the CPI-U to the C-CPI-U. For comparison and robustness, three methods are used, each giving qualitatively similar results. The methods are described in turn below. Figure 1 gives an overview of the methods.

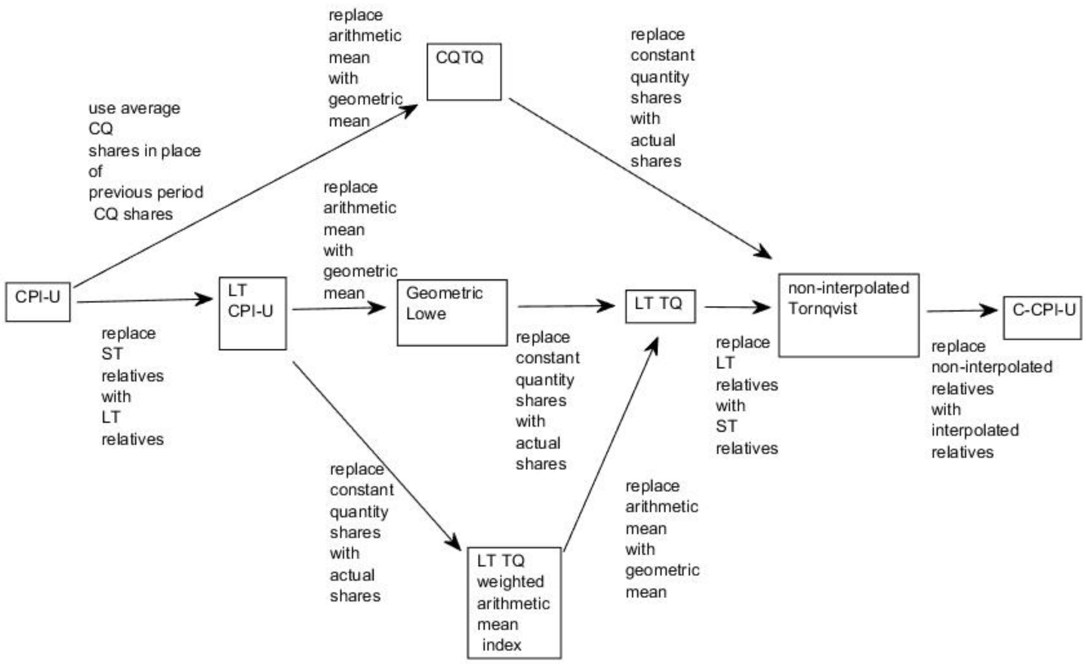
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Figure 1

The first updates the weights first, and the second and third change the formula first.

The intermediate steps must be meaningful in themselves so that the results can be interpreted in a meaningful way. If not, the differences would be irrelevant to explaining the difference between the CPI-U and C-CPI-U. If the C-CPI-U is a relevant approximation of a COLI, this means each change should be designed to move a COGI to a COLI.

While any change to an index is likely to change the month-to-month movements, the differences in month-to-month changes are small enough that it is difficult to see the significance of any particular change just looking at those movements. Only when the monthly changes are cumulated over time to make the index levels do obvious differences become visible. Therefore, instead of graphing monthly changes, this study focuses on the differences in the index levels.

The period studied is the ten years covering December 1999 through December 2009. The C-CPI-U began to be published in January 2002, and because the final values are published with a two year lag, the first month of the index was January 2000, which used the index relative from December 1999 to January 2000. Thus the first month of data used is December 1999, month 0, for which the indexes are normalized to 1, so the reported indexes begin in January 2000, which is month 1 out of 120. CPI data is used, which has the expenditures, index levels, and index relatives needed to construct the indexes.

The results are summarized in Table 1, and described in detail below. To give an overview, Figure 2 graphs the index levels for the CPI-U, C-CPI-U, and the intermediate indexes of changing the weights first. The main feature to note is that the indexes are in two clumps, around the CPI-U and the C-CPI-U, and the gap in the middle is the effect of using the Tornqvist weights. None of the other changes are very significant on this scale.

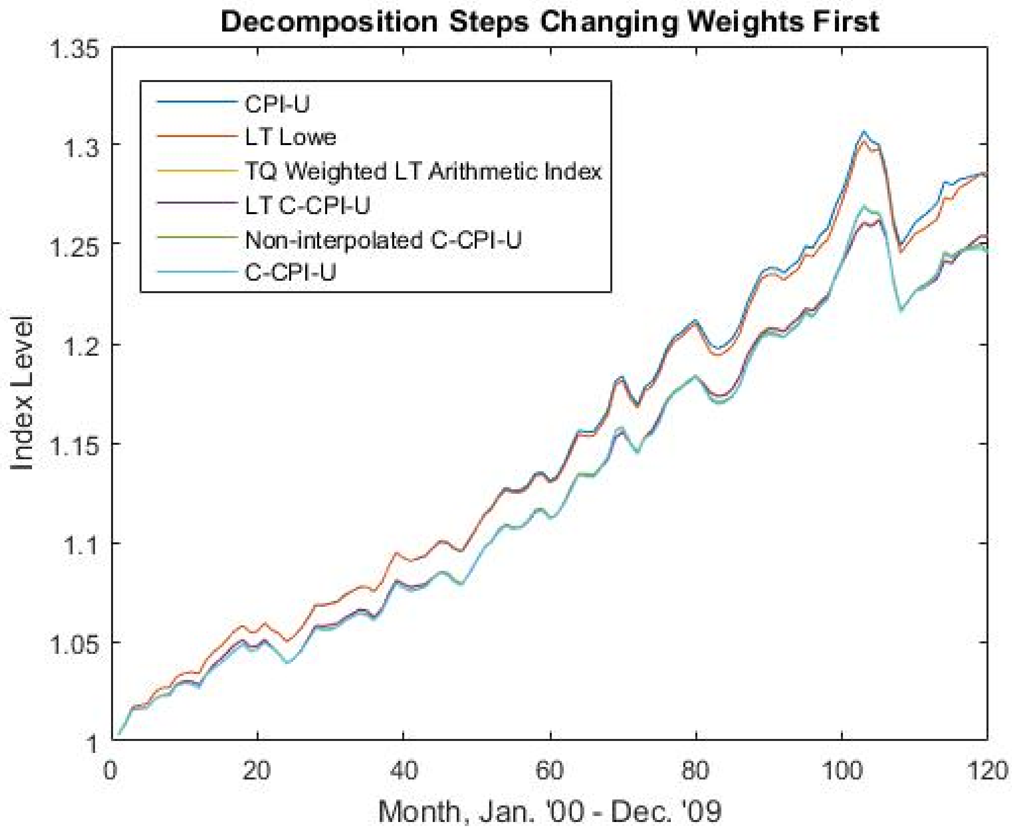

Figure 2

Table 1: Effects of each Step by Method of Breakdown as % of Total CPI-U vs. C-CPI-U Difference

	Method		
	Weights First	Formula First - Geomeans	Formula First - CQTQ
Replace ST Relatives with LT Relatives	8.38%	Same	NA
Weight Effect: Replace Constant Quantity Shares with Actual Shares	85.98%	86.25%	96.22%
Formula Effect: Replace Arithmetic Mean with Geometric Mean	2.05%	1.79%	1.45%
Replace LT Relatives with ST Relatives	1.25%	Same	NA
Replace Non-interpolated Relatives with Interpolated Relatives	2.34%	Same	Same

4 Changing Weights First

The movement from a COGI towards a COLI from quantity weight updating, and thus from effectively more frequent chaining, is not necessarily direct, or monotonic. Changing the weights every month in an intermediate step can cause a spurious correlation between the weights and index relatives that would not occur in the C-CPI-U. This would make the index drift up or down meaninglessly, an effect referred to as 'chain drift', as described by Szulc (1983). It is similar to the problem of formula bias described in Reinsdorf (1998a). The problem of chain drift in the intermediate indexes is described in detail in the Appendix, section 8.1.

To see how much weight updating moves the CPI-U to an approximate COLI, the drift effect must be taken out first. One suggested method is used by Ivancic, Diewert, and Fox (2011). They use scanner data to construct various price indexes at quarterly, monthly, and weekly levels, and find large chain drift. They suggest using a method of combining longer spanning indexes to create drift free indexes, called the GEKS method.

This study uses a different method based on long term price relatives that is more direct and intuitive for upper level CPI aggregation. The goal is to remove the short run correlation between the price relatives and implicit quantity weights without changing the index levels or the effects of long term consumer substitution. Only the long run price information matters for a direct index. Without the effects of drift, using the current expenditure shares which don't hold quantities constant will approximate a continuously chained Laspeyres index.

The modified index series must show the same inflation rate for each item-area up to any given month, so that it yields the same index level. It should also be able to closely replicate the CPI-U. Otherwise, it has substantively inaccurate changes, and has incorrect inflation information for that month. Also, each item-area must have the same unchanging price relative for each month, so that there is no correlation between relatives and weights. Then there is no drift for any given weights, and thus no drift pattern which would be different for different index formulas. Therefore, denoting the modified index level in month τ used to match the actual index level for month t as $I_{ia\tau}^{LT,t}$, and the modified relative used to match the actual index level for month t as d_{iat}, for item i in area a,

$$\frac{I_{ial}^{LT}}{I_{i,a,l-1}^{LT}} = \frac{I_{iak}^{LT}}{I_{i,a,k-1}^{LT}} = d_{iat} \tag{7}$$

for any two months $l, k < t$.

Since each item-area index level is the first month's level, denoted by I_{ia0} for the level in Dec. '99, multiplied by all the intervening index relatives,

$$I_{iat} = I_{ia0}\Pi_{\tau=1}^{t} R_{ia\tau} \tag{8}$$

, the only d_{iat} that would satisfy these conditions is

$$d_{iat} = \left(\frac{I_{iat}}{I_{ia0}} \right)^{\frac{1}{t}} \tag{9}$$

and thus

$$I_{iat}^{LT,t} = I_{ia0} d_{iat}^{t} \tag{10}$$

. Therefore the modified index level would equal the actual index level for month t, but not necessarily for month $\tau < t$.

The unique solution to these constraints is to use the long term relatives of all the item-area price relatives for each given month. This is the same as the unweighted geometric mean of index relative from month 1 to t,

$$d_{iat} = \Pi_{\tau=1}^{t} R_{ia\tau}^{\frac{1}{t}} \tag{11}$$

. It is also the only price information that a direct index between month 1 and t would use.

Since the Tornqvist has little chain drift, the chain drift cannot be causing much of the difference between the CPI-U and C-CPI-U. The short term oscillations cause the drift, so removing them cannot have much of an effect on the CPI-U vs. C-CPI-U difference.

The chained modified index should be the same as the chained Laspeyres index if the chaining is frequent enough. That way, the only difference will be due to the frequency of chaining. In other words it will only differ by chain drift, which is what should be taken out.

The chained long term Laspeyres index approaches the chained Laspeyres as the interval shrinks, if the expenditure shares are constant, which in turn equals a Divisia index. This is shown in Theorem 1.

Theorem 1 *When the expenditure share weights are constant, $s_{ia\tau} = s_{ia.}$ for all τ, a continuously chained Laspeyres index using item-area index relatives is equal to a continuously chained Laspeyres index using the long term relatives defined in* (9) *and* (11),

$$\exp\left(\int_{\tau=0}^{t} \ln\left(\Sigma_{ia} s_{ia.} R_{ia\tau}\right) d\tau \right) = \exp\left(\int_{\tau=0}^{t} \ln\left(\Sigma_{ia} s_{ia.} d_{ia\tau}\right) d\tau \right) \tag{12}$$

.

Proof. *In Appendix 5.2.* ■

As shown in Appendix 5.1, the expenditure shares can be treated as constant, because if the C-CPI-U or other indexes hold the shares constant, the resulting index changes very little compared to the total CPI-U vs. C-CPI-U difference.

The long term relatives d_{iat} are different for each month t, given the initial month 0. Each d_{iat} therefore defines a different series of index levels for each item i in area a tracing a smooth inflation path from 0 to t. From (10), for a given month t, the smooth path of long term index levels is given by

$$I^{LT,t}_{ia\tau} = I_{ia0} d^{\tau}_{iat} \tag{13}$$

for the long term index level at month τ leading from 0 to t. These series over all items and areas are then used to create a different all-items national index series from 0 to t. The final month of each series is then used as the level for the long term overall index for month t.

The Lowe index made with the long term relatives, or long term Lowe index, is then

$$\Pi^t_{\tau=1} \frac{\Sigma_{ia} Aggweight_{iaB} I^{LT,t}_{ia\tau}}{\Sigma_{ia} Aggweight_{iaB} I^{LT,t}_{i,a,\tau-1}} = \Pi^t_{\tau=1} \Sigma_{ia} \frac{Aggweight_{iaB} I^{LT,t}_{i,a,\tau-1}}{\Sigma_{ia} Aggweight_{iaB} I^{LT,t}_{i,a,\tau-1}} d_{iat} \tag{14}$$

. The long term Lowe index is in fact a direct index over the two year period that the weights don't change. After that, it is as direct as the Lowe index: it is the same as chaining every two years. In fact, they are exactly the same for the first two years before the first CPI-U weight update in the data, as seen in Figure 2. This is not surprising since the Lowe index is circular (the same as a direct Lowe index from the first to last months) over each two year period in which the quantity weights do not change.[8] The only difference between the long term Lowe index and the CPI-U is due to the correlations between the long run relatives and the biennial weight updates which is slightly different than the correlations between the short term relatives.

[8] The chained Lowe is

$$\Pi^t_{\tau=1} \frac{\Sigma_{ia} Aggweight_{iaB} I_{iat}}{\Sigma_{ia} Aggweight_{iaB} I_{i,a,t-1}} =$$

$$\frac{\Sigma_{ia} Aggweight_{iaB} I_{ia1}}{\Sigma_{ia} Aggweight_{iaB} I_{ia0}} \cdot \frac{\Sigma_{ia} Aggweight_{iaB} I_{ia2}}{\Sigma_{ia} Aggweight_{iaB} I_{ia1}} \tag{15}$$

$$\cdot \frac{\Sigma_{ia} Aggweight_{iaB} I_{ia3}}{\Sigma_{ia} Aggweight_{iaB} I_{ia2}} \cdots \frac{\Sigma_{ia} Aggweight_{iaB} I_{iat}}{\Sigma_{ia} Aggweight_{iaB} I_{i,a,t-1}} \tag{16}$$

$$= \frac{\Sigma_{ia} Aggweight_{iaB} I_{iat}}{\Sigma_{ia} Aggweight_{iaB} I_{ia0}} \tag{17}$$

over the period that the $Aggweight_{iaB}$ doesn't change. The chained long term Lowe is

$$\Pi^t_{\tau=1} \frac{\Sigma_{ia} Aggweight_{iaB} d^{\tau}_{iat}}{\Sigma_{ia} Aggweight_{iaB} d^{\tau-1}_{iat}} =$$

$$\frac{\Sigma_{ia} Aggweight_{iaB} d^1_{iat}}{\Sigma_{ia} Aggweight_{iaB} d^0_{iat}} \cdot \frac{\Sigma_{ia} Aggweight_{iaB} d^2_{iat}}{\Sigma_{ia} Aggweight_{iaB} d^1_{iat}} \tag{18}$$

$$\cdot \frac{\Sigma_{ia} Aggweight_{iaB} d^3_{iat}}{\Sigma_{ia} Aggweight_{iaB} d^2_{iat}} \cdots \frac{\Sigma_{ia} Aggweight_{iaB} d^t_{iat}}{\Sigma_{ia} Aggweight_{iaB} d^{t-1}_{iat}} \tag{19}$$

$$= \frac{\Sigma_{ia} Aggweight_{iaB} d^t_{iat}}{\Sigma_{ia} Aggweight_{iaB} d^0_{iat}} \tag{20}$$

over that same period.

The first step in the breakdown by changing weights first is to take the total difference between the CPI-U and the long term Lowe index, from month 1 to T = 120, Jan. '00 to Dec. '09 respectively, given by

$$\Sigma_{t=1}^{T}\left[\Pi_{\tau=1}^{t}\Sigma_{ia}\frac{Aggweight_{iaB}I_{i,a,\tau-1}}{\Sigma_{ia}Aggweight_{iaB}I_{i,a,\tau-1}}R_{ia\tau}-\Pi_{\tau=1}^{t}\Sigma_{ia}\frac{Aggweight_{iaB}I_{i,a,\tau-1}^{LT,t}}{\Sigma_{ia}Aggweight_{iaB}I_{i,a,\tau-1}^{LT,t}}d_{iat}\right] \tag{21}$$

. This is 8.38% of the total difference between the CPI-U and C-CPI-U, given by

$$\Sigma_{t=1}^{T}\left[\Pi_{\tau=1}^{t}\Sigma_{ia}\frac{Aggweight_{iaB}I_{i,a,\tau-1}}{\Sigma_{ia}Aggweight_{iaB}I_{i,a,\tau-1}}R_{ia\tau}-\Pi_{\tau=1}^{t}\Pi_{ia}R_{ia\tau}^{s_{ia\tau}^{TQ}}\right] \tag{22}$$

, where the Tornqvist mean shares are given by

$$s_{ia\tau}^{TQ}=\frac{s_{i,a,\tau-1}+s_{ia\tau}}{2} \tag{23}$$

. As mentioned, the difference for the first two years is zero.

The second step is to use the Tornqvist weights in place of the Lowe weights, to see the effect of implicit weight updating free of drift, which is the only change from the long term Lowe index.

Because the Tornqvist weights are an average of the current month's expenditure share and the previous month's share, the weights contain implicit quantity information from both months. This is a property the Tornqvist weighted long term arithmetic index shares with superlative indexes such as the Fisher and Walsh indexes, and pseudo-superlative indexes such as the Marshall-Edgeworth index. Because superlative indexes tend to give similar results, it raises the question of whether simply using the Tornqvist weights may have given the index a superlative-like quality and explain why it is so close to the C-CPI-U without using a superlative formula.

However, a chained long term Laspeyres index, which uses monthly updated expenditure share weights from the previous month only and not the current, is very close to the Tornqvist weighted long term arithmetic index, with a total difference of 1.41% of the difference between the CPI-U and C-CPI-U. This can be explained by the fact that the share trends make little difference, so using a lagged share vs. an average with the current share would make little difference. Also, as will be shown below with the Constant Quantity Tornqvist, using an average weight instead of the previous month's weight makes little difference when quantities are held constant.

The Tornqvist weighted long term arithmetic index is

$$\Pi_{\tau=1}^{t}\Sigma_{ia}s_{ia\tau}^{TQ}d_{iat} \tag{24}$$

.

The difference between the long term Lowe index and the Tornqvist weighted long term arithmetic index,

$$\Sigma_{t=1}^{T}\left[\Pi_{\tau=1}^{t}\Sigma_{ia}\frac{Aggweight_{iaB}I_{i,a,\tau-1}^{LT,t}}{\Sigma_{ia}Aggweight_{iaB}I_{i,a,\tau-1}^{LT,t}}d_{iat}-\Pi_{\tau=1}^{t}\Sigma_{ia}s_{ia\tau}^{TQ}d_{iat}\right] \tag{25}$$

is 85.98% of the total difference between the CPI-U and C-CPI-U.

The third step is to change the arithmetic mean to a geometric mean, making the long term Tornqvist index,

$$\Pi_{\tau=1}^{t}\Pi_{ia}d_{iat}^{s_{ia\tau}^{TQ}} \tag{26}$$

. The difference between the Tornqvist weighted long term arithmetic index and the long term Tornqvist,

$$\Sigma_{t=1}^{T}\left[\Pi_{\tau=1}^{t}\Sigma_{ia}s_{ia\tau}^{TQ}d_{iat}-\Pi_{\tau=1}^{t}\Pi_{ia}d_{iat}^{s_{ia\tau}^{TQ}}\right] \tag{27}$$

is only 2.05% of the total difference. As seen in Figure 2, this is a very small difference relative to the overall CPI-U to C-CPI-U difference. This is consistent with geometric and arithmetic means being approximately equal as interval the index is chained at shrinks. If one month is a short enough interval, a small difference is what would be expected.

Another difference between the CPI-U and C-CPI-U is that the CPI-U uses the actual item-area index relatives, while the C-CPI-U uses interpolated relatives. Many item-area cells are only priced bimonthly. For those cells, the index relative is 1 and the index doesn't change for the unpriced months. For use in the C-CPI-U, however, the unpriced months are given the square root of the next priced index relative, for both the priced and unpriced months, smoothing the inflation over the two months. The long term relatives were made with the actual non-interpolated index relatives.

The fourth step, from the long term Tornqvist to the Tornqvist made with non-interpolated relatives, adds back the short run variation in the item-area index relatives. This yields the difference that using the long term relatives makes for the Tornqvist. The difference is

$$\Sigma_{t=1}^{T}\left[\Pi_{\tau=1}^{t}\Pi_{ia}d_{iat}^{s_{ia\tau}^{TQ}}-\Pi_{\tau=1}^{t}\Pi_{ia}R_{ia\tau}^{s_{ia\tau}^{TQ}}\right] \tag{28}$$

which comes to only 1.25% of the total difference. It is even less of an effect than the difference between the CPI-U and the long term Lowe index. Using the long term relatives, which holds the inflation rate constant, doesn't bias the C-CPI-U. This is because the trends in the shares matter little, so it makes little difference when the inflation in a certain item-area occurred when using expenditure shares in an index – all the relatives would be weighted about the same. This is similar to the effect of a long term Lowe index, since by construction the CPI-U holds the quantity weights constant most of the time. In fact, if the shares are constant, the long term Tornqvist is exactly the same as the normal Tornqvist, as shown by Theorem 2.

Theorem 2 *When the expenditure share weights are constant, $s_{ia\tau}^{TQ}=\frac{1}{2}\left(s_{i,a,\tau-1}+s_{ia\tau}\right)=s_{ia.}$ for all τ, a Tornqvist index using item-area index relatives is equal to a Tornqvist index using the long term relatives defined in* (9) *and* (11)*,*

$$\exp\left(\Sigma_{t=1}^{T}\Sigma_{ia}s_{ia.}\ln R_{iat}\right)=\exp\left(\Sigma_{t=1}^{T}\Sigma_{ia}s_{ia.}\ln d_{iat}\right) \tag{29}$$

.

Proof. In Appendix 5.2. ■

Finally, the fifth step is to move from the interpolated relatives Tornqvist to the actual C-CPI-U. Denoting the interpolated index relative for item i in area a for month τ as $Rint_{ia\tau}$, the difference is

$$\Sigma_{t=1}^{T}\left[\Pi_{\tau=1}^{t}\Pi_{ia}R_{ia\tau}^{s_{ia\tau}^{TQ}}-\Pi_{\tau=1}^{t}\Pi_{ia}Rint_{ia\tau}^{s_{ia\tau}^{TQ}}\right] \tag{30}$$

which is 2.34% of the total difference.

If the first and fourth steps are considered part of the formula effect, the total formula effect is 8.38%+2.05%+1.25% = about 11.86% of the total, while the weight updating effect, step 2, is 85.98% of the total, which is the large majority. The remainder is the effect of using interpolated index relatives.

5 Changing Formula First

Another set of valid intermediate steps that move from the CPI-U to the C-CPI-U involves changing the formula first, and then the weights. The first step is now to change from an arithmetic mean to a geometric mean, keeping the share weights the same as in equation (2).

However, this change would suffer from a different kind of drift. Consider a geometric Lowe index, the log of which is

$$\begin{aligned}\ln G_t^{LoCQ} &= \Sigma_{ia}\frac{Aggweight_{iaB}I_{i,a,\tau-1}}{\Sigma_{ia}Aggweight_{iaB}I_{i,a,\tau-1}}\ln R_{ia\tau}\\ &= \Sigma_{ia}\frac{Aggweight_{iaB}I_{i,a,\tau-1}}{\Sigma_{ia}Aggweight_{iaB}I_{i,a,\tau-1}}\ln\left(\frac{I_{iat}}{I_{i,a,t-1}}\right)\end{aligned} \tag{31}$$

. This uses the Lowe $I_{i,a,t-1}$ term in both the share and in the index relative. This means that when there is price bouncing and the previous index level is high, the relative will be low. Of quantities move inelastically in price, the share will be high, and vice versa. This creates a negative correlation between the weights and the relatives, which causes negative drift (if quantities moved elasticially with price, the drift would be positive). This can be seen in Figure 3, where the geometric Lowe is below even the C-CPI-U.

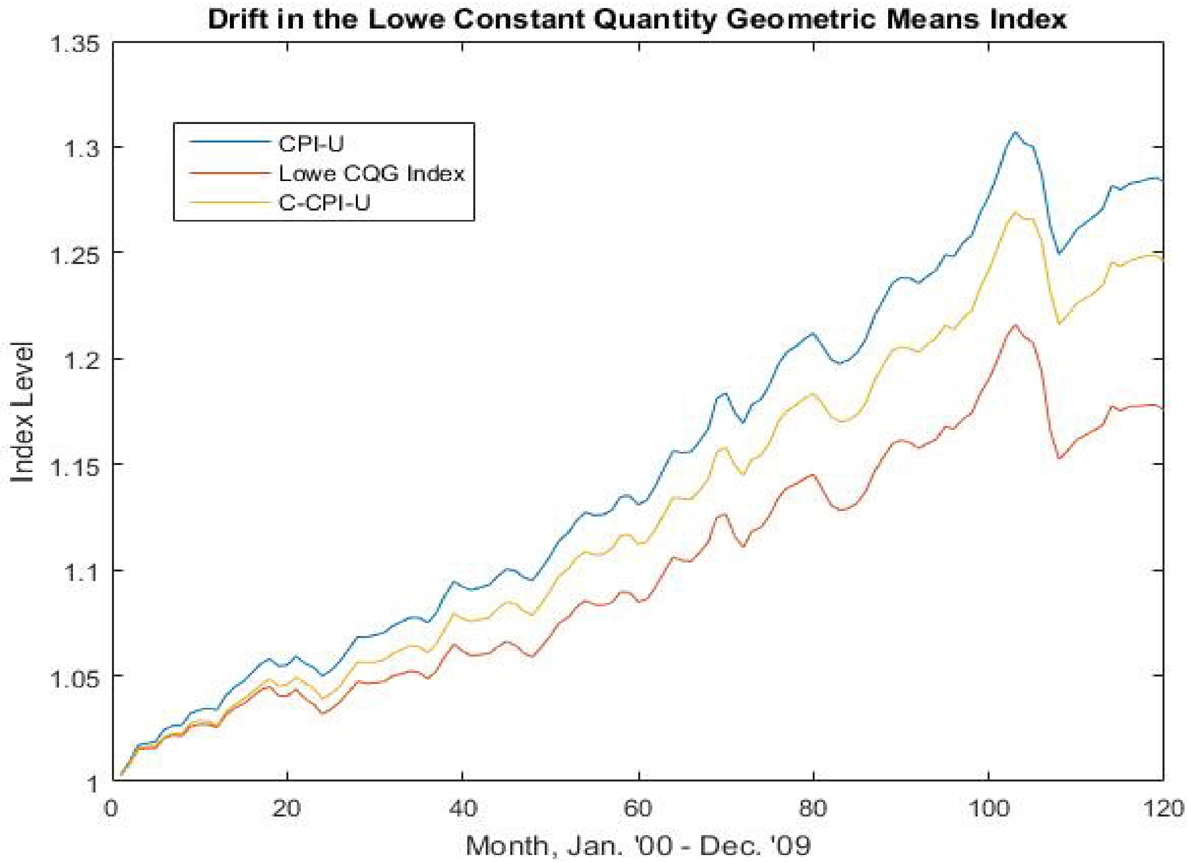

Figure 3

To get a meaningful breakdown, a long term Lowe constant quantity geometric index (LT Lowe CQG) must be constructed

$$G_{t-1,t}^{LTCQ} = \exp\left(\Sigma_{ia} \frac{Aggweight_{iaB} I_{i,a,\tau-1}^{LT,t}}{\Sigma_{ia} Aggweight_{iaB} I_{i,a,\tau-1}^{LT,t}} \ln d_{iat} \right) \qquad (32)$$

.

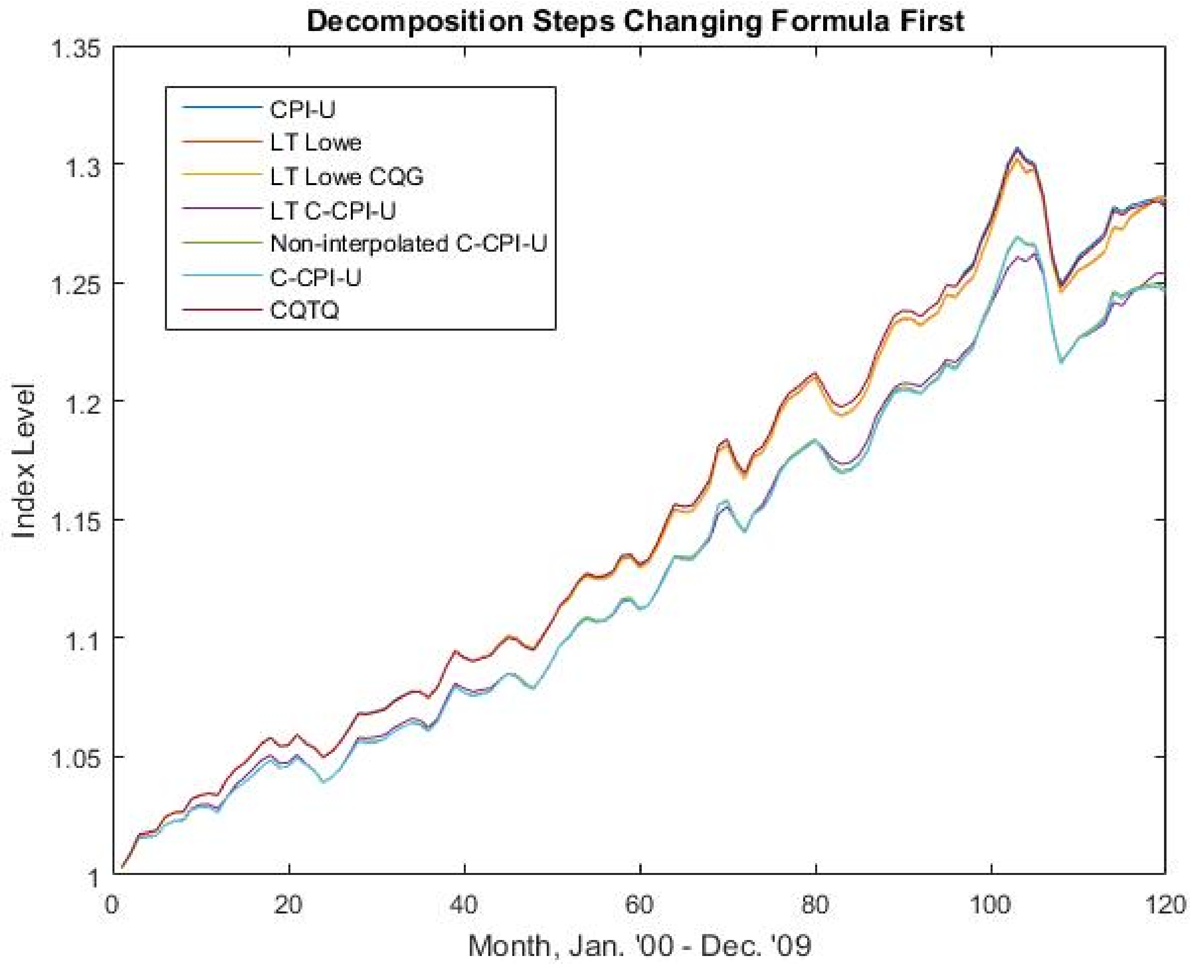

Figure 4

The long term Lowe constant quantity geometric mean index is graphed in Figure 4. There is still a little drift from first month, since for the first month in each series used to make the monthly index levels, if the previous index level is high, the month's relative will be low but the share will be high.

An alternative is to smooth out the weights in adjacent months by using the Tornqvist weights. Greenlees (2013) shows how the Lowe arithmetic mean can be approximated with a Tornqvist-like geometric mean which holds the quantities fixed, as in the Lowe index. This Constant Quantity Tornqvist (CQTQ) is simply the Tornqvist index with constant quantities in each period imposed. Greenlees (2014) showed that the CQTQ is in fact a second order logarithmic approximation to a Lowe or Laspeyres index. It uses the Lowe weight updating method, but takes an average of the Lowe shares over the current and previous months, like the Tornqvist index. The log CQTQ is

$$\ln T_{t-1,t}^{LoCQ} = \Sigma_{ia} \frac{1}{2} s_{i,a,B} \left(\frac{R_{i,a,B,t-1}}{L_{B,t-1}} + \frac{R_{i,a,B,t}}{L_{B,t}} \right) \ln R_{iat} \tag{33}$$

, where $R_{i,a,B,t}$ denotes the item-area index relative from the base period B to month t, and $L_{B,t}$ denotes the Laspeyres index of inflation from base period B to month t. The CQTQ updates the base period expenditure share $s_{i,a,B}$ for item i in area a by the relative inflation each item-area up to months t-1 and t, thus holding the implicit quantity constant.

If the index relatives bounce as in Figure 5 in Appendix 5.1 at the points A-B-F-G-H-I-J, so that a relatively high price is followed by a relatively low price, the effect on the first share will be the opposite of the effect on the second, so there will not be much correlation with the index relative. This can be seen in Figure 4, as the CQTQ is almost on top of the CPI-U, with a difference of 1.45% of the total.

Therefore there are two ways to do the first step.

One is to move to the long term Lowe index first. This is the same as step 1 before, with a 8.38% difference with the CPI-U.

The second step is to go to the long term Geometric Lowe index, for a difference of 1.79%.

The third step is move to the long term C-CPI-U by inserting the Tornqvist share weights in place of the constant-quantity share weights, for a difference of 86.25%.

The last step two steps are the same as steps 4 and five above, for 1.25% and 2.34% of the difference respectively.

The other way to change formula first is to move directly to the CQTQ first. As one might expect, this makes little difference. In fact, this is only 1.45% of the difference, as mentioned above, similar to step 2 above and step 3 when changing the weights first.

Using this sequence of steps, the next step is to move to the non-interpolated Tornqvist, by using the Tornqvist share weights and allowing the implicit quantities to change, for a whopping 96.22% of the total difference. The final step is the same as the final steps above, for 2.34%. In the first two methods, the moves to and from using the long term relatives summed to 8.38%+1.25% = 9. 63%. This is roughly the difference between the weight effects for the first and third methods.

The results are robust to either of the three methods used, as in each case the large majority of the total difference is due to using share weights that allow the implicit quantities to change over long periods of time. The results are also robust to whether the weights or the formula is changed first. In both cases, allowing the implicit quantities to change and using long term relatives creates an index that is very close to the C-CPI-U. In fact the last method, moving to the CQTQ and then to the C-CPI-U, shows that the result that the majority of the difference is due to allowing quantity changes doesn't depend on using long term index relatives at all.

All of these results were generated using Dec. '99 as the initial period, and all the long term relatives were defined as the path from Dec. '99 to the current month according to (9). Therefore it is important to check whether the initial month affects the results. The same breakdowns were calculated using every month in the first 6 years of data, Dec. '99 - Dec. '05, as initial periods. The months used for each index went from the initial month to the same final month, Dec. '09. Months later than Dec. '05 were not used because the total length of the index would be too short for reliable results.[9] The simple means of the fractions listed in Table 1 over all 72 of the initial periods are reported in Table 2. The results are similar to Table 1.

Table 2: Mean Effects Across Initial Periods of each Step by Method of Breakdown as % of Total CPI-U vs. C-CPI-U Difference

	Method		
	Weights First	Formula First - Geomeans	Formula First - CQTQ
Replace ST Relatives with LT Relatives	6.29%	Same	NA
Weight Effect: Replace Constant Quantity Shares with Actual Shares	88.10%	88.33%	94.04%
Formula Effect: Replace Arithmetic Mean with Geometric Mean	3.64%	3.41%	2.82%
Replace LT Relatives with ST Relatives	-1.18%	Same	NA
Replace Non-interpolated Relatives with Interpolated Relatives	3.14%	Same	Same

The mean effects of the long term relatives is actually negative, at -1.18%. This effect varies between positive and negative for different initial months, and different lengths of time averaged over, but is always very small. Effectively, the long term C-CPI-U is on average the same as the C-CPI-U, only differing by small noise. Since the TQ weighted long term arithmetic index only differs from the long term C-CPI-U by the formula effect, around 3.5%, this means the TQ weighted long term arithmetic index is almost the same as the C-CPI-U (not counting the effects of interpolated relatives, which are not necessarily part of a COLI). That means that simply chaining at a monthly frequency and using long term relatives is 96.5% sufficient for measuring a COLI, without any formula assumptions. If such an index was used in place of the C-CPI-U, for many purposes it would be sufficient, providing a measure of a COLI while avoiding any theoretical issues surrounding it.

[9] Fewer index months meant more variance from single month shocks, since the index levels are cumulative monthly changes. Since the index levels were more variable and there were fewer months to sum the differences in index levels, the results became unstable and unreliable.

6 Conclusions/Discussion

All three approaches come to the same conclusion. The large majority of the difference between the CPI-U and the C-CPI-U is due to the weighting differences, which constrain quantities to be constant over long periods in the CPI-U but allow implicit quantities to change over long periods in the C-CPI-U. One month is a short enough period so that holding quantity or share weights constant makes little difference, but the effects add up if the quantity weights are held constant for an average of 36 months.

While the formula assumption may be important when very close precision is required, if the index is chained at the monthly level, long term price relatives and weights updated without holding quantities constant are sufficient for an index to effectively approximate to the C-CPI-U relative to the CPI-U. Therefore, the fact that the C-CPI-U is lower than the CPI-U does not depend on assumptions that are often criticized. In fact, if the assumptions used for approximating a COLI are undesirable, it isn't necessary to try to try to measure a COLI at all to get effectively the same results.

A large economics literature has debated the appropriate formula to use in measuring cost-of-living-indexes. Every index formula in significant use is either an arithmetic mean, geometric mean, or some combination thereof. If chain drift is controlled for, such as by using long term relatives, and if chaining is at the monthly frequency, it makes little difference using CPI data whether an arithmetic or geometric mean is used for CPI data over the time period studied, because prices change little over a single month and the price relatives are close to 1. Therefore, this study provides an example of how the formula may not be that relevant for many purposes.

Chaining an index more frequently is basically interpolating within the range of the data. This is filling in the space between the current and previous period. Of course, interpolations converge as the range shrinks. It's not extrapolating outside of the range of the data, where the functional choice matters.

This study develops a new method to control for chain drift. The fact that the item-area expenditure shares can be effectively held constant for geometric mean index construction makes the long term relatives an effective solution to chain drift. For other data, a different technique may be required. However, this can still provide an example of a general method that could be modified for another environment.

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8 Appendices

8.1 Chain Drift

As noted above, both the CPI-U and the C-CPI-U are chained indexes.[10] The reason for using a chained index instead of a direct index between the first and last periods is that in a direct index, the weights on the first period become less and less relevant to the present as the time span lengthens. But a potential drawback to chaining is that short run price movements that are correlated with the weights can cause a chained index to drift upwards or downwards in ways that do not reflect trend inflation. In fact, overall inflation may be higher for a chained index than even the inflation rate of the highest inflation item. Certain chained indexes that had constant weights would, however, be circular, or the same as a direct index, since in that case there would be no correlation between weights and price movements.[11]

Drift in a Laspeyres index is caused by price oscillations, or 'bouncing', and was described by Szulc (1983). As an example, consider the following index relative formula, given by

$$L_{t-1,t}^{constshare} = \Sigma_i s_i R_{it} \tag{34}$$

over items i between period $t-1$ and t. This is a form of a chained Laspeyres index which implicitly changes the quantity weights each period such that the expenditure shares remain constant over time. Suppose there are two goods with equal expenditure shares when the weights are set, so that $s_1 = s_2$. Also suppose each good's price bounces between 1 and 2 every period, so that each

[10] See CPI Manual (ILO 2004), chapters 9, 15, and 19.

[11] For example, if the Lowe index never updated its weights at all, multiplying successive periods' inflation rates together would mean the numerator for each period would equal and divide out the denominator from the past period, and the result would simply be the direct index between the first and last periods.

price relative is either 2 or $\frac{1}{2}$. There would be no long run inflation for either good. Yet this index relative would give an inflation rate of $\frac{1}{2}(2 + \frac{1}{2}) = 1.25$, or 25% inflation every period.

As another example, consider Figure 5, which graphs a possible common pattern of continuous price change for an item-area in the CPI.

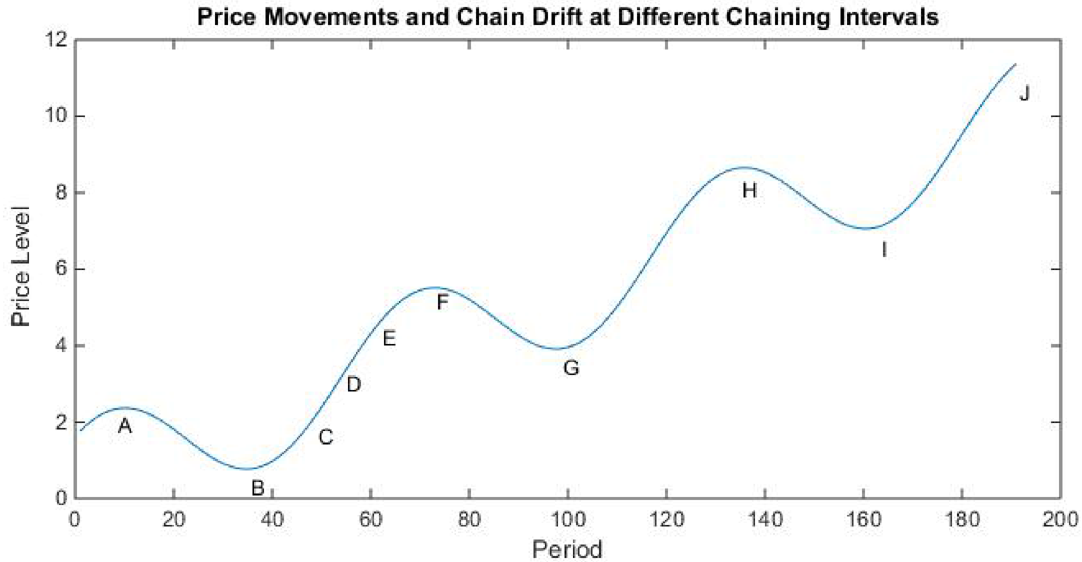

Figure 5

In Figure 5, the price level tends to rise in the long run, due to overall inflation: but the price level also goes up and down in the short run. If the index is chained at a long interval, so that one index period is from A to J, the short run variation matters little. Likewise, if the index is chained at a frequency such that the chaining points are B-G-I, or A-F-H-J, then the short run variation will not influence the long run movement of the index. But consider if the index was chained at the frequency such that the chaining points were A-B-F-G-H-I-J, which are the peaks and troughs over the price movements. At A, the price is relatively high, so consumers will buy relatively less of it, and the quantity will be relatively low. At B, the reverse is true, and the quantity will be relatively high. A Laspeyres index chained at this frequency will put a low weight on the price decline from A to B, but a high weight on the price rise from B to F. Because of overweighting the price increases relative to declines, it could give a higher overall inflation rate attributed to that item than its long run inflation of A to J.

One example of an item-area cell that could have this pattern is Rice, Pasta, Cornmeal, and Other Cereal Products in Philadelphia, in Figure 6. Each point in Figure 6 is a month. If the monthly frequency corresponded to the A-B-F-G-H-I-J frequency in Figure 5, the resulting chaining points could look like this.[12]

[12] Index level series for item-area cells such as this have a small sample size and are not published by the BLS.

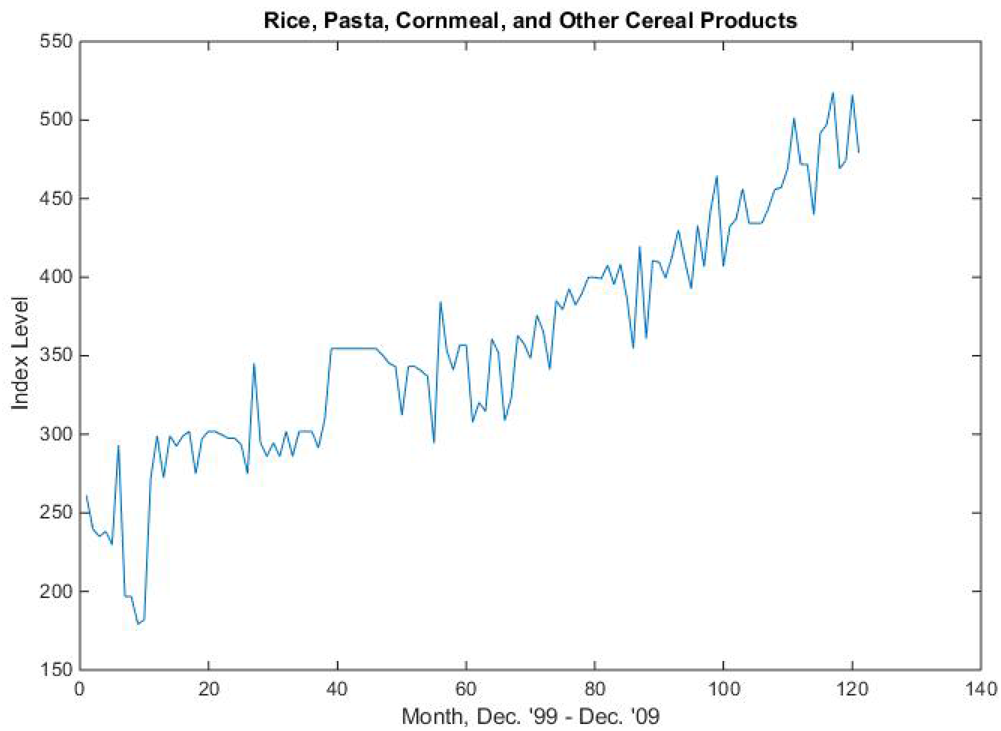

Figure 6

However, if the frequency of chaining increased further so that the chaining points are points like B-C-D-E-F-etc., the oscillations will once again not matter as much. If it was continuously chained, the oscillations wouldn't matter at all.

Thus, only if the index was chained close enough to the A-B-F-G-H-I-J frequency do the oscillations matter. Szulc (1983) used the Tacoma Narrows Bridge, which collapsed in 1940, as an analogy to this resonance effect. At a certain chaining frequency, a chained Laspeyres could diverge from the continuous Laspeyres.

A chained Laspeyres index is, in a way, a contradiction. A Laspeyres index measures the change in the cost of buying last period's fixed basket, but a chained Laspeyres changes the basket every period. As a measure of the change in the cost of living, it assumes quantities are fixed but then changes quantities. It opens up the possibility of a garbage-in-garbage-out result that doesn't necessarily mean anything. This could outweigh any gain of a chained index in estimating a COLI over using a direct index.

Indeed, garbage-in-garbage-out is what actually happens if the Laspeyres arithmetic mean of index relatives is used with the monthly updated or Tornqvist share weights in place of the Lowe constant quantity share weights. The Chained Laspeyres index relative is

$$L^C_{t-1,t} = \Sigma_{ia} s_{i,a,t-1} R_{iat} \tag{35}$$

. A Tornqvist weighted chained arithmetic mean index relative, which uses the TQ mean expenditure shares in a chained Laspeyres index in place of a single month's previous period share, is

$$L^{TQ}_{t-1,t} = \Sigma_{ia} s^{TQ}_{iat} R_{iat} \tag{36}$$

where $s^{TQ}_{iat} = \frac{s_{iat}+s_{i,a,t-1}}{2}$. As seen in Figure 7, the Chained Laspeyres is higher than the CPI-U by 102% of the distance between the CPI-U and C-CPI-U, while the Tornqvist weighted chained arithmetic mean index is higher than the CPI-U by 148% of the distance.

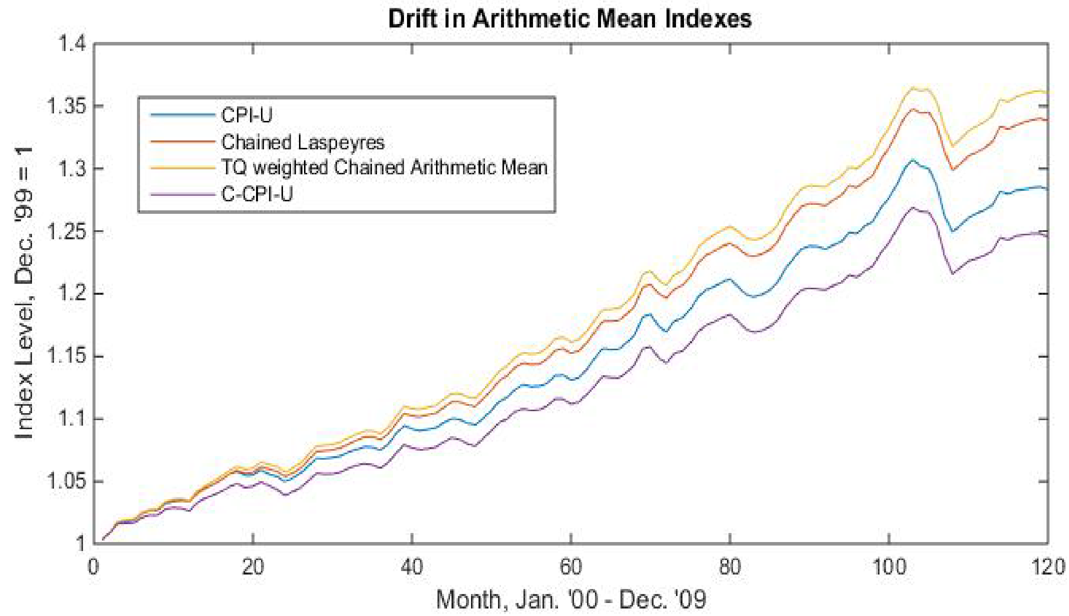

Figure 7

8.1.1 Expenditure Shares Can be Treated as Constant

The reason for this pattern is that holding the actual shares constant matters little for index construction, at least on the scale of comparing the CPI-U to the C-CPI-U. This means that they can be considered as effectively constant for this purpose. Cage, Greenlees, and Jackman in fact report that the shares do not have large changes over time. Therefore, using updated share weights is not very different from using the index relative in (34).

To demonstrate how holding the shares constant makes little difference, Figure 8 compares the C-CPI-U to a geometric means index where the shares are held constant at the average share over the ten years for each item-area. The only difference between these indexes is that the C-CPI-U uses an average share between the current and previous month. The two indexes are very close relative to the CPI-U, showing that holding the shares constant or allowing them to change makes little difference in an index. While Greenlees and Williams (2009) do find that shares change in response to relative index level changes, these effects are clearly not large for the purpose of comparing the CPI-U to the C-CPI-U.

The initial estimates of the C-CPI-U also demonstrate this. The final C-CPI-U had only been released with a two year lag due to the processing time for the CE. However, from the outset, initial estimates have been published that used expenditure shares from the same base period as the CPI-U base periods, and so held the shares constant over the same period that the CPI-U's quantity weights are held constant. When the C-CPI-U was first planned, it was thought that there would be a significant downward bias in the initial C-CPI-U relatives compared to the final C-CPI-U relatives.[13] This difference was to be measured over time as the average ratio between the initial and final C-CPI-U relatives. This adjustment factor would then be used to adjust the initial estimates to make them closer to the final relatives. However, when the adjustment factor turned out to be variable and small enough to omit, it was decided to drop the adjustment factor altogether.[14]

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[13] See Cage, Greenlees, and Jackman (2002).

[14] Assuming that the expenditure shares are constant can still make a significant difference with the final C-CPI-U values for some purposes. The BLS has now changed from a constant shares geometric mean index for the initial estimates to a more flexible constant-elasticity-of-substitution function to make the initial values even closer to the final ones. See Klick (2016).

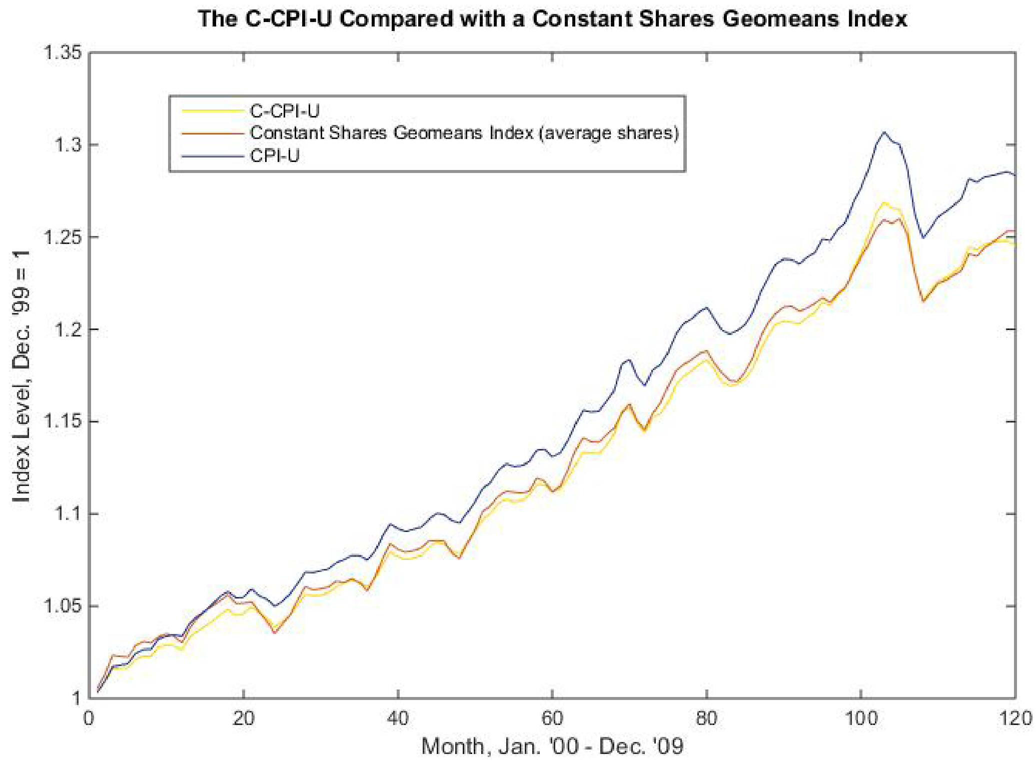

Figure 8

Except for sampling variation, this implies that the implicit quantity can be treated as moving roughly inversely proportionate to price, so that the shares change little. The fact that the TQ weights are averages of two month's shares doesn't help if shares don't trend much. In fact it makes them even closer to being constant, which even is closer the index in (34), and thus the Tornqvist weighted chained arithmetic mean index has even more upward chain drift than the Chained Laspeyres.[15]

Drift in a geometric mean index, where the index is an expenditure share weighted geometric mean of item-area index relatives, is not caused by the same data pattern that causes drift in an arithmetic mean index. Instead of being caused by a correlation between quantities and index relatives, geometric mean index chain drift is caused by a correlation between the expenditure shares and the index relatives. Suppose that the expenditure share weight rose when the price rose, so that for a pattern like Figure 5, the expenditure share weights at

[15] The response of shares is prices is actually inelastic, shown in Greenlees & Williams (2009). Therefore, quantities move less than if the shares were constant, causing the drift pattern to be more pronounced when the average share is used.

points A, F, and H were relatively high, while for points B, G, and I they were relatively low. If a geometric mean index was chained at the intervals of A-B-F-G-H-I-J, it would have a high weight when the index relatives were falling, and vice versa when rising. Therefore it would have downward chain drift. Because the Tornqvist index uses an average expenditure share from the current and previous month, there is usually little drift in it. The short run variation in the item-area index relatives is only slightly correlated with the monthly Tornqvist weights, so that using a geometric mean removes the drift that the Chained Laspeyres has.

Therefore, the effect of formula on the chained Laspeyres index or TQ weighted Chained Arithmetic Mean index is not the same as on the CPI-U.

8.2 Equivalence of Continuously Chained Laspeyres and Long Term Laspeyres

To show this, first it will be shown that for continuous chaining, an arithmetic mean of price relatives is the same as the geometric mean. This is also shown in Diewert (1980). It is a well-known approximation to a logarithm that $\ln R \cong R-1$, and $exp(R-1) \cong R$. Since all price relatives are close to 1 as the chaining interval is small enough,[16] this means that at a given period in time, the geometric mean of relatives weighted by w_{ia} such that $\Sigma_{ia} w_{ia} = 1$ is

$$\begin{aligned} \exp\left(\Sigma_{ia} w_{ia} \ln R_{iat}\right) &= \exp\left(\Sigma_{ia} w_{ia}\left(R_{iat}-1\right)\right) = \exp\left(\Sigma_{ia}\left(w_{ia} R_{iat} - w_{ia}\right)\right) \\ &= \exp\left(\Sigma_{ia} w_{ia} R_{iat} - 1\right) = \Sigma_{ia} w_{ia} R_{iat} \end{aligned} \tag{37}$$

which is the arithmetic mean.

Theorem 1 shows that an Laspeyres index using the long term relatives approaches the same limit with continuous chaining as a Laspeyres with the month index relatives. Therefore, using the long term relatives can avoid the spurious upward chain drift and approach the Divisia limit without diverging first as chaining becomes more frequent.

Proof. Proof of Theorem 1.

Because an arithmetic and geometric mean are interchangeable at continuous

[16] Actual price changes are typically in discrete jumps, so at many points in time the relatives would not actually equal one. However, since these price changes would not ocurr at the same time, it can be assumed that the mass of price changes is small enough so that the relatives are not significantly different from one.

chaining,

$$
\begin{aligned}
\exp\left(\int_{\tau=0}^{t} \ln\left(\Sigma_{ia} s_{ia\tau} R_{ia\tau}\right) d\tau\right) &= \exp\left(\int_{\tau=0}^{t} \ln\left(\exp\left(\Sigma_{ia} s_{ia\tau} \ln R_{ia\tau}\right)\right) d\tau\right) \\
&= \exp\left(\int_{\tau=0}^{t} \left(\Sigma_{ia} s_{ia\tau} \ln R_{ia\tau}\right) d\tau\right) \\
&= \exp\left(\Sigma_{ia} \int_{\tau=0}^{t} s_{ia\tau} \ln R_{ia\tau} d\tau\right) \qquad (38)
\end{aligned}
$$

. With constant shares,

$$
\begin{aligned}
&= \exp\left(\Sigma_{ia} \int_{\tau=0}^{t} s_{ia.} \ln R_{ia\tau} d\tau\right) \qquad (39) \\
&= \exp\left(\Sigma_{ia} s_{ia.} \int_{\tau=0}^{t} \ln R_{ia\tau} d\tau\right) \qquad (40)
\end{aligned}
$$

. Similarly for the index using long term relatives,

$$
\begin{aligned}
\exp\left(\int_{\tau=0}^{t} \ln\left(\Sigma_{ia} s_{ia.} d_{ia\tau}\right) d\tau\right) &= \exp\left(\Sigma_{ia} s_{ia.} \int_{\tau=0}^{t} \ln d_{iat} d\tau\right) \qquad (41) \\
&= \exp\left(\Sigma_{ia} s_{ia.} \ln d_{iat} \int_{\tau=0}^{t} d\tau\right) \\
&= \exp\left(\Sigma_{ia} s_{ia.} t \ln d_{iat}\right)
\end{aligned}
$$

. The continuous version of (11) is

$$
d_{iat} = \exp\left(\frac{1}{t} \int_{\tau=0}^{t} \ln R_{iat} d\tau\right) \qquad (42)
$$

. Plugging this into the last line of (41) yields (39). ■

This is the case illustrated in Figure 5, where the chaining points move to B-C-D-E-F-etc., and then even closer together. Therefore both indexes are a chained long term index. Also, as mentioned, it has been shown that the continuous Laspeyres is equal to the Divisia index, since the Divisia index is simply a geometric mean version of a continuous Laspeyres.

Theorem 2 gives an analogous result for the Tornqvist index, but without requiring continuous chaining.

Proof. Proof of Theorem 2.

Rewriting the r.h.s. of (29) by switching the order of addition and bringing out the shares as

$$\exp\left(\Sigma_{ia} s_{ia.} \Sigma_{t=1}^{T} \ln d_{iat}\right)$$

and plugging in (11) yields

$$\begin{aligned} & \exp\left(\Sigma_{ia} s_{ia.} \Sigma_{t=1}^{T} \Sigma_{t=1}^{T} \frac{1}{T} \ln R_{iat}\right) \\ = \quad & \exp\left(\Sigma_{ia} s_{ia.} \Sigma_{t=1}^{T} \ln R_{iat}\right) \end{aligned}$$

which can again be rewritten as the l.h.s. of (29) by again switching the order of addition and bringing out the shares. ■