Combining Price Indices in Temporal Hierarchies*

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Abstract

There is a growing demand from central banks, governments, banks, real estate developers, and households for higher frequency house price indices. Such indices are not widely available since they are considered less reliable. In this paper we show that by combining lower and higher frequency indices (e.g., annual, quarterly and monthly) it is possible to improve quality at all frequencies. Furthermore, our method provides more timely indices. For example, rather than having to wait until the end of the year to obtain a new annual index, or the end of a quarter for a new quarterly index, our method produces a new annual and quarterly index every month. While the method can be applied to price indices in any field, in our empirical application we focus specifically on house price indices. We show that our reconciled annual, quarterly and monthly house price indices are more reliable than their unreconciled counterparts. Improving both reliability and timeliness allows users to make more informed decisions. (***JEL***. C33; C43; R31)

Keywords: Housing market; Hedonic estimation; Higher frequency indices; Temporal hierarchy; Kalman filter; Real-time reconciliation

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1 Introduction

In recent years there is an increased interest in higher frequency price indices. For example, since the global financial crisis central banks have become more aware of how developments in the housing market can affect the rest of the economy, and in some cases threaten financial stability. Hence there is a need for more timely data, thus allowing central banks and other regulatory bodies to respond faster to undesirable developments in the housing market, other asset markets, and more generally to macroeconomic variables such as inflation.

Higher frequency indices are typically constructed from less data, and hence can be less reliable than lower frequency indices (see Hill et al., 2020a). Also, higher and lower frequency indices can show different trends, and hence be inconsistent with each other. By combining indices defined over different frequencies (e.g., annual, quarterly and monthly), we show how it is possible to improve the reliability of indices of all frequencies, while simultaneously generating new annual quarterly and monthly indices each time a new month becomes available.

Our starting point is price indices arranged in temporal hierarchies. In such a hierarchy, the basic building blocks are the time periods over which the highest frequency index is defined (e.g., monthly). The second highest frequency consists of a whole number of highest frequency periods (e.g., three months). The next frequency consists of a whole number of periods from the previous layer in the hierarchy (e.g., four quarters), etc.

Rather than viewing any inconsistencies that arise between indices of different frequencies as a nuisance, we show how this provides an opportunity to improve the quality of the price indices at all frequencies. With this in mind, we propose here a least-squares method that reconciles price indices arranged in temporal hierarchies. It follows that all the indices are adjusted in the reconciliation process.

Our method is related to a least-squares reconciliation approach for temporal hierarchies developed by Athanasopoulos et al. (2017), which in turn draws on Hyndman et al. (2011), and Hyndman et al. (2016). There is, however, an important difference between our approach and that of Athanasopoulos et al. (2017). Athanasopoulos et al. focus on data series that can be summed across time periods. In other words, it is assumed that the annual value of a series in a particular year should equal the sum of the quarterly series in that same year.

Instead of assuming that the price indices in log form are additive over the temporal hierarchy, we impose different identifying restrictions. Our approach draws on ideas from the multilateral price index literature, and especially the Gini-Eltetö-Szulc (GEKS) method (see, for example, Diewert, 1999, and Balk, 2008). The key is to formulate different combinations of indices that provide alternative answers to the same question. Our identifying restrictions then entail requiring after reconciliation that these different indices asking the same question give the same answer.

We explore a number of variants on the basic method. Our variants differ along two dimensions. First we reconcile indices at the lowest frequency (here annual). We consider three ways of doing this, ordinary-least squares (OLS), weighted least squares (WLS) and a Kalman filter approach. We then consider six different methods for backing out reconciled higher frequency indices from the reconciled annual indices. Each backing-out method can be used in combination with any of the OLS, WLS or Kalman filter methods from the first stage. Four of the backing-out methods are subject to revisions whenever a new month is added to the dataset. However, the last two backing-out methods are not subject to revisions. These two, the rolling-window (RW) and the system recursive methods also perform best in our empirical application according to our quality metric.

In addition to producing improved indices at all frequencies, these indices are also produced in real time. For example, rather than having to wait until the end of the year to obtain a new annual index, or the end of a quarter for a new quarterly index, our method produces a new annual and quarterly index every month. The improvement in both reliability and timeliness will allow users to make more informed decisions.

The remainder of the paper is structured as follows. Section 2 develops our least-squares reconciliation method as it applies to two and three layer temporal hierarchies. Section 3 interprets the reconciled price indices obtained from three layer hierarchies. Section 4 explores the link between our least-squares reconciliation method for temporal hierarchies and the GEKS multilateral price index method. Section 5 considers weighted variants on our basic method. Section 6 shows how the method can be extended to produce real-time reconciled indices at all frequencies every period. Section 7 develops a metric for evaluating the performance of indices. Section 8 provides an empirical application, using house price indices computed at a monthly, quarterly and annual frequency. Our main findings are summarized in Section 9.

2 Reconciling Temporal Hierarchies of Price Indices

2.1 The simplest case

The simplest case of a temporal hierarchy of price indices is where there are two layers, and the higher frequency is double that of the lower frequency. Here we focus on the case where the lower frequency is annual and the higher frequency biannual. The reconciliation is done at the level of the lowest frequency index, which is here annual. We reconcile each pair of adjacent years separately.

In this simplest case we have three distinct price indices defined on an annual time horizon. Let $P_{1,2}$ denote the price change from year 1 and 2. $P_{11,12}$ the price change from the first half of year 1 to the first half of year 2, and $P_{21,22}$ the price change from the second half of year 1 to the second half of year 2. Taking the geometric mean of $P_{11,21}$ and $P_{12,22}$, we obtain an alternative measure to $P_{1,2}$ of the price change from year 1 to year 2.

Our objective is to alter the original indices $P_{1,2}$, $P_{11,21}$ and $P_{12,22}$ by the logarithmic-least-squares amount necessary to reconcile our two annualized indices. Reconciliation here means ensuring that the following condition is satisfied: $\ln \hat{P}_{1,2} = 0.5(\ln \hat{P}_{11,21} + \ln \hat{P}_{12,22})$.

This least-squares problem can be formulated as follows:

$$\text{Min}_{\ln \hat{P}_{1,2}, \ln \hat{P}_{11,21}, \ln \hat{P}_{12,22}} \left[(\ln \hat{P}_{1,2} - \ln P_{1,2})^2 + 0.5(\ln \hat{P}_{11,21} + \ln \hat{P}_{12,22} - \ln P_{11,21} - \ln P_{12,22})^2 \right],$$

$$\text{such that} \quad \ln \hat{P}_{1,2} = 0.5(\ln \hat{P}_{11,21} + \ln \hat{P}_{12,22}). \tag{1}$$

We can rewrite this problem more compactly in matrix notation as follows:

$$y = S\beta + \varepsilon \tag{2}$$

where

$$S = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad y = \begin{pmatrix} \ln P_{1,2} \\ 0.5(\ln P_{11,21}) \\ 0.5(\ln P_{12,22}) \end{pmatrix}$$

and ε is an error vector representing the aggregation error with zero mean and covariance matrix Σ. Hyndman et al. (2011, 2016) proposed a variant on this linear model in the context of reconciliation of forecasts. They showed that when the aggregation errors approximately satisfy the same aggregation structure as the original data, then OLS and GLS estimates of β are identical. Even if the aggregation errors do not satisfy this assumption, they argue the OLS solution will still be a consistent way of reconciling the base forecast.

We will first consider the least-squares case as this allows us to study this reconciliation approach in the context of price changes and relate it to the multilateral price index literature. In the least-squares case the projection matrix of the reconciliation is

$$S(S'S)^{-1}S' = \begin{pmatrix} 2/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & -1/3 \\ 1/3 & -1/3 & 2/3 \end{pmatrix}.$$

The least-squares solution is now given by:

$$\hat{y} = S\hat{\beta} = S(S'S)^{-1}S'y, \tag{3}$$

where

$$\hat{y} = \begin{pmatrix} \ln \hat{P}_{1,2} \\ 0.5(\ln \hat{P}_{11,21}) \\ 0.5(\ln \hat{P}_{12,22}) \end{pmatrix} = \begin{pmatrix} \hat{\beta}_1 + \hat{\beta}_2 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2\ln P_{1,2} + \frac{1}{2}(\ln P_{11,21} + \ln P_{12,22}) \\ \ln P_{11,21} + \frac{1}{2}(2\ln P_{1,2} - \ln P_{12,22}) \\ \ln P_{12,22} + \frac{1}{2}(2\ln P_{1,2} - \ln P_{11,21}) \end{pmatrix}. \tag{4}$$

It follows from (4) that

$$\ln \hat{P}_{1,2} = \frac{1}{3}\left[2\ln P_{1,2} + \frac{1}{2}(\ln P_{11,21} + \ln P_{12,22})\right], \tag{5}$$

$$\ln \hat{P}_{11,21} = \frac{1}{3}\left[2\ln P_{11,21} + (2\ln P_{1,2} - \ln P_{12,22})\right], \tag{6}$$

$$\ln \hat{P}_{12,22} = \frac{1}{3}\left[2\ln P_{12,22} + (2\ln P_{1,2} - \ln P_{11,21})\right]. \tag{7}$$

From these equations in each case we can interpret the reconciled price index as a weighted geometric mean of the direct unreconciled index and the indirect unreconciled index, where the direct index is given twice the weight of the indirect index.

2.2 Three layer hierarchies

Consider now the case of a temporal hierarchy consisting of annual, biannual, and quarterly price indices.[1] Focusing on the reconciliation of years 1 and 2, we now have the following additional indices defined on an annual time interval: $P_{1q1,2q1}$ compares the first quarters of years 1 and 2, $P_{1q2,2q2}$ compares the second quarters of years 1 and 2, etc.

Now we have three reconciliation equations:
(i) $\hat{P}_{1,2} = (\hat{P}_{1q1,2q1} \times \hat{P}_{1q2,2q2} \times \hat{P}_{1q3,2q3} \times \hat{P}_{1q4,2q4})^{1/4}$
(ii) $(\hat{P}_{11,21}) = (\hat{P}_{1q1,2q1} \times \hat{P}_{1q2,2q2})^{1/2}$
(iii) $(\hat{P}_{12,22}) = (\hat{P}_{1q3,2q3} \times \hat{P}_{1q4,2q4})^{1/2}$

Three more equations relating the reconciled prices indices can be derived from (i), (ii) and (iii). These are the following:
(iv) $\hat{P}_{1,2} = (\hat{P}_{11,21} \times \hat{P}_{12,22})^{1/2}$.
(v) $\hat{P}_{1,2} = [(\hat{P}_{1q1,2q1} \times \hat{P}_{1q2,2q2})^{1/2} \times \hat{P}_{12,22}]^{1/2}$.
(vi) $\hat{P}_{1,2} = [\hat{P}_{11,21} \times (\hat{P}_{1q3,2q3} \times \hat{P}_{1q4,2q4})^{1/2}]^{1/2}$.

Our objective is to alter the unreconciled price indices by the logarithmic least squares amount necessary so that (i), (ii), and (iii) are satisfied. This reconciliation problem can be formulated in matrix notation as follows:

$$y = S\beta + \varepsilon,$$

[1] In our empirical application we focus on the case of a three-layer hierarchy consisting of years, quarters and months. However, for ease of exposition we describe here a hierarchy consisting of years, half-years, and quarters.

where

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad y = \begin{pmatrix} \ln P_{1,2} \\ 0.5(\ln P_{11,21}) \\ 0.5(\ln P_{12,22}) \\ 0.25(\ln P_{1q1,2q1}) \\ 0.25(\ln P_{1q2,2q2}) \\ 0.25(\ln P_{1q3,2q3}) \\ 0.25(\ln P_{1q4,2q4}) \end{pmatrix}, \tag{8}$$

and ε again denotes an error vector.

3 Interpreting Reconciled Price Indices in Three Layer Hierarchies

It is informative to consider how the reconciled price indices are formed by taking linear combinations of the original unreconciled indices. In this regard we will focus on the three layer hierarchy of yearly, biannual, and quarterly indices. In this case, the matrix $S(S'S)^{-1}S'$ takes the following form:

$$S(S'S)^{-1}S' = \frac{1}{21}\begin{pmatrix} 12 & 6 & 6 & 3 & 3 & 3 & 3 \\ 6 & 10 & -4 & 5 & 5 & -2 & -2 \\ 6 & -4 & 10 & -2 & -2 & 5 & 5 \\ 3 & 5 & -2 & 13 & -8 & -1 & -1 \\ 3 & 5 & -2 & -8 & 13 & -1 & -1 \\ 3 & -2 & 5 & -1 & -1 & 13 & -8 \\ 3 & -2 & 5 & -1 & -1 & -8 & 13 \end{pmatrix} \tag{9}$$

Solutions for the reconciled annual price indices as functions of the original price indices are obtained by inserting (9) and the y vector in (8) into (3).

$$\ln \hat{P}_{1,2} = \frac{1}{21}\left\{12\ln P_{1,2} + 6\left[\frac{1}{2}(\ln P_{11,21} + \ln P_{12,22})\right]\right.$$

$$\left.+3\left[\frac{1}{4}(\ln P_{1q1,2q1} + \ln P_{1q2,2q2} + \ln P_{1q3,2q3} + \ln P_{1q4,2q4})\right]\right\}. \tag{10}$$

It can be seen that the annual reconciled index is a weighted geometric mean of three competing unreconciled annualized indices. The direct unreconciled annual index $P_{1,2}$ gets a weight of 12/21. From reconciliation equation (iv), the indirect annual index obtained by taking the geometric mean of the two annualized biannual indices gets a weight of 6/21. Finally, substituting the reconciliation equations (ii) and (iii) into (iv), the indirect annual index obtained by taking the geometric mean of the four annualized quarterly indices gets a weight of 3/21.

The solution for $P_{11,21}$ is:

$$\ln \hat{P}_{11,21} = \frac{1}{21}\left\{12\ln P_{1,2} + 10\ln P_{11,21} - 4\ln P_{12,22} + \frac{5}{2}\ln P_{1q1,2q1} + \frac{5}{2}\ln P_{1q2,2q2}\right.$$

$$\left. - \ln P_{1q3,2q3} - \ln P_{1q4,2q4}.\right\} \quad (11)$$

This solution for $P_{11,21}$ can be reinterpreted as follows:

$$\ln \hat{P}_{11,21} = \frac{1}{21}\left\{10\ln P_{11,21} + 4(2\ln P_{1,2} - \ln P_{12,22})\right.$$

$$\left. +2\left[2\ln P_{1,2} - \frac{1}{2}\left(\ln P_{1q3,2q3} + \ln P_{1q4,2q4}\right)\right] + 5\left[\frac{1}{2}(\ln P_{1q1,2q1} + \ln P_{1q2,2q2})\right]\right\}. \quad (12)$$

Here the reconciled index comparing the first half of year 1 with the first half of year 2 is again written as a weighted geometric mean of competing unreconciled indices answering the same question. The unreconciled direct annualized biannual index $P_{11,21}$ gets a weight of 10/21, while the indirect index combining $P_{1q1,2q1}$ and $P_{1q2,2q2}$ gets a weight of 5/21, which is half that of the direct index. Thus far the pattern is analogous to the cases considered above. However, there are two more indirect indexes that also contribute to the solution for the reconciled index. These are the indirect indexes formed by combining $P_{1,2}$ and $P_{11,12}$, which gets a weight of 4/21, and the indirect index combining $P_{1,2}$, $P_{1q3,2q3}$ and $P_{1q4,2q4}$, which gets a weight of 2/21.

The solution for $P_{1q1,2q1}$ is:

$$\ln \hat{P}_{1q1,2q1} = \frac{1}{21}\{12\ln P_{1,2} + 10\ln P_{11,21} - 4\ln P_{12,22} + 13\ln P_{1q1,2q1} - 8\ln P_{1q2,2q2}$$

$$- \ln P_{1q3,2q3} - \ln P_{1q4,2q4}.\} \quad (13)$$

This solution can likewise be reinterpreted as a weighted geometric mean of competing unreconciled indices answering the same question:

$$\ln \hat{P}_{1q1,2q1} = \frac{1}{21}\{13\ln P_{1q1,2q1} + 5(2\ln P_{11,21} - \ln P_{1q2,2q2})$$

$$+2(4\ln P_{1,2} - 2\ln P_{12,22} - \ln P_{1q2,2q2}) + (4\ln P_{1,2} - \ln P_{1q2,2q2} - \ln P_{1q3,2q3} - \ln P_{1q4,2q4})\}. \quad (14)$$

Now the unreconciled direct annualized quarterly index $P_{1q1,2q1}$ gets a weight of 13/21, the indirect index formed by combining $P_{11,21}$ and $P_{1q2,2q2}$ gets a weight of 5/21, the indirect index formed by combining $P_{1,2}$, $P_{12,22}$, and $P_{1q2,2q2}$ gets a weight of 2/21, and the indirect index combining $P_{1,2}$, $P_{1q2,2q2}$, $P_{1q3,2q3}$ and $P_{1q4,2q4}$ gets a weight of 1/21.

To discern the underlying structure it is useful to note that there are seven unknowns to be estimated: $\hat{P}_{1q1,2q1}$, $\hat{P}_{1q2,2q2}$, $\hat{P}_{1q3,2q3}$, $\hat{P}_{1q4,2q4}$, $\hat{P}_{11,21}$, $\hat{P}_{12,22}$, $\hat{P}_{1,2}$. In addition there are three constraints relating these seven unknowns, given by the reconciliation equations (i), (ii) and (iii) above. Hence there are only four degrees of freedom. For example, given values for $\hat{P}_{1q1,2q1}$, $\hat{P}_{1q2,2q2}$, $\hat{P}_{1q3,2q3}$, and $\hat{P}_{1q4,2q4}$, then $\hat{P}_{1,2}$, $\hat{P}_{11,21}$ and $\hat{P}_{12,22}$ can be derived from equations (i), (ii) and (iii), respectively.

There are 35 possible ways of combining four out of seven variables, i.e., $7!/(3! \times 4!)$. Of these only 21 retain the four degrees of freedom (i.e., they do not contain redundancies). For example, equations (v) and (vi) imply that the combinations $(\hat{P}_{1,2}, \hat{P}_{1q1,2q1}, \hat{P}_{1q2,2q2}, \hat{P}_{12,22})$, and $(\hat{P}_{1,2}, \hat{P}_{11,21}, \hat{P}_{1q3,2q3}, \hat{P}_{1q4,2q4})$ each have only three degrees of freedom, and hence cannot recover values for the missing three variables.

The 21 combinations of four variables that are sufficient to derive the other variables are listed below:

1. $P_{1q1,2q1}, P_{1q2,2q2}, P_{1q3,2q3}, P_{1q4,2q4}$
2. $P_{11,21}, P_{1q2,2q2}, P_{1q3,2q3}, P_{1q4,2q4}$
3. $P_{11,21}, P_{1q1,2q1}, P_{1q3,2q3}, P_{1q4,2q4}$
4. $P_{12,22}, P_{1q1,2q1}, P_{1q2,2q2}, P_{1q4,2q4}$
5. $P_{12,22}, P_{1q1,2q1}, P_{1q2,2q2}, P_{1q3,2q3}$
6. $P_{11,21}, P_{12,22}, P_{1q1,2q1}, P_{1q3,2q3}$
7. $P_{11,21}, P_{12,22}, P_{1q1,2q1}, P_{1q4,2q4}$
8. $P_{11,21}, P_{12,22}, P_{1q2,2q2}, P_{1q3,2q3}$
9. $P_{11,21}, P_{12,22}, P_{1q2,2q2}, P_{1q4,2q4}$
10. $P_{1,2}, P_{1q1,2q1}, P_{1q2,2q2}, P_{1q3,2q3}$
11. $P_{1,2}, P_{1q1,2q1}, P_{1q2,2q2}, P_{1q4,2q4}$
12. $P_{1,2}, P_{1q1,2q1}, P_{1q3,2q3}, P_{1q4,2q4}$
13. $P_{1,2}, P_{1q2,2q2}, P_{1q3,2q3}, P_{1q4,2q4}$
14. $P_{1,2}, P_{11,21}, P_{1q1,2q1}, P_{1q3,2q3}$
15. $P_{1,2}, P_{11,21}, P_{1q1,2q1}, P_{1q4,2q4}$
16. $P_{1,2}, P_{11,21}, P_{1q2,2q2}, P_{1q3,2q3}$
17. $P_{1,2}, P_{11,21}, P_{1q2,2q2}, P_{1q4,2q4}$
18. $P_{1,2}, P_{12,22}, P_{1q1,2q1}, P_{1q3,2q3}$
19. $P_{1,2}, P_{12,22}, P_{1q1,2q1}, P_{1q4,2q4}$
20. $P_{1,2}, P_{12,22}, P_{1q2,2q2}, P_{1q3,2q3}$
21. $P_{1,2}, P_{12,22}, P_{1q2,2q2}, P_{1q4,2q4}$

Each of these combinations provides a different way of constructing temporally reconciled indices from four unreconciled indices. The solution to the least squares reconciliation problem can be interpreted as the geometric mean of these 21 combinations.

For example, consider the case of $\ln \hat{P}_{1q1,2q1}$. As noted there are only four chaining paths for estimating $\ln \hat{P}_{1q1,2q1}$ given the available indices:

A. $\ln P_{1q1,2q1}$

B. $2 \ln P_{11,21} - \ln P_{1q2,2q2}$

C. $4 \ln P_{1,2} - 2 \ln P_{12,22} - \ln P_{1q2,2q2}$

D. $4 \ln P_{1,2} - \ln P_{1q2,2q2} - \ln P_{1q3,2q3} - \ln P_{1q4,2q4}$

The combinations that include each of these chain paths are as follows:

A. 1, 3, 4, 5, 6, 7, 10, 11, 12, 14, 15, 18, 19

B. 2, 8, 9, 16, 17

C. 20, 21

D. 13

Thus it can readily be seen that the A approach appears 13 times, the B approach 5 times, C 2 times and D just 1 time. These weights correspond exactly with those in (14).

4 An Analogy with the Multilateral Price Index Literature

There is an interesting parallel here with the GEKS method used to transitivize price indices in the multilateral price index literature (see, for example, Diewert, 1999, and Balk, 2008). In the GEKS context there are no temporal hierarchies. Rather, the GEKS method takes a set of intransitive bilateral price indices and alters them by the logarithmic least squares amount necessary to make them transitive (or reconciled using our terminology).

Algebraically, this least squares problem can be written as follows:

$$\min_{\ln P_j, \ln P_k} \sum_{j=1}^{I} \sum_{k=1}^{I} (\ln P_k - \ln P_j - \ln P_{j,k})^2, \tag{15}$$

where I is the number of countries participating in a multilateral comparison, $P_{j,k}$ denotes the observed bilateral price index comparison between countries j and k, P_k denotes a multilateral (reconciled) price index for country k, and the normalization $P_1 = 1$ is imposed. The solutions, $\hat{\ln P_j}, \hat{\ln P_k}$ are the ordinary least squares (OLS) estimators of $\ln P_j, \ln P_k$ in the regression model:

$$\ln P_{j,k} = \ln P_k - \ln P_j + \epsilon_{j,k}, \tag{16}$$

where $\epsilon_{j,k}$ is a random error term.

The GEKS price indices take the following form:

$$\frac{P_k^{GEKS}}{P_j^{GEKS}} = \exp\left(\ln\hat{} P_k - \ln\hat{} P_j\right) = \prod_{i=1}^{I} (P_{j,i} \times P_{i,k})^{1/I} = (P_{j,k})^{2/I} \prod_{i \neq j,k}^{I} (P_{j,i} \times P_{i,k})^{1/I}, \tag{17}$$

where P_k^{GEKS} denotes the GEKS price index for country k, and $i = 1, \ldots, I$ indices the countries included in the multilateral comparison.[2]

As can be seen from (17), the GEKS solution for a pair of countries j and k gives twice the weight to the direct bilateral comparison between j and k, as to all the indirect comparisons (each of which involves chaining via a third country i). This finding is reminiscent of our result for two layer hierarchies in (5). An intriguing parallel also exists with the result for our three layer hierarchy in (10). Here the reconciled annual comparison gives the indirect comparison made using biannual indices half the weight as the direct comparison, while the indirect comparison using quarterly indices gets quarter the weight.

5 Weighted Reconciliation in Three Layer Hierarchies

5.1 The basic case of weighted reconciliation

Hyndman et al. (2016) discuss the optimally reconciled forecasts as those given by the generalised least squares (GLS) solution,

$$\hat{y} = S\tilde{\beta} = S(S'\Sigma^{\dagger}S)^{-1}S'\Sigma^{\dagger}y, \tag{18}$$

where, $\Sigma^{\dagger}$ is the generalised inverse of the covariance matrix of ε in the model in (2). However, $\Sigma^{\dagger}$ is unknown and virtually impossible to estimate. An alternative might be to use weighted least squares (WLS). That is, replacing $\Sigma^{\dagger}$ by W, a diagonal matrix with elements equal to the inverse of the variances of the elements of ε,

$$\hat{y}^{WLS} = S\tilde{\beta} = S(S'WS)^{-1}S'Wy, \tag{19}$$

When information is available on the number of observations over which an index was constructed, this information can be used in the reconciliation process. For example, the empirical comparison in Section 8 focuses on house price indices for the Eastern Suburbs of Sydney, Australia. There is a strong seasonal cycle in the number of transactions in Sydney. Very few transactions occur over the summer months, from the second half of December through to the end of February. As a consequence the number of transactions

[2] We have assumed in (17) that the bilateral price index formula $P_{j,k}$ satisfies the country reversal test (i.e., $P_{j,k} = 1/P_{k,j}$). All superlative price indices satisfy this test (see Diewert, 1976).

is always lower in the first quarter of the year. The effect is even more dramatic when the highest frequency index is monthly. Continuing with the three layer example from Section 3 where the highest frequency is the quarterly index, an implication of the seasonal cycle in transactions is that the annualized quarterly index $P_{1q1,2q1}$ may tend to be less reliable than the other annualized quarterly indices $P_{1q2,2q2}$, $P_{1q3,2q3}$ and $P_{1q4,2q4}$. In order to capture this reliability effect, we propose to use a weighted least squares estimation with weights proportional to the number of transactions. Let n_{1q1} denote the number of transactions in year 1, quarter 1, n_{11} the number of transactions in the first half of year 1, and n_1 the number of transactions in year 1. For the three-layer hierarchy of annual, biannual and quarterly indices, then we define the inverse of the covariance matrix of ε as follows,

$$W_{TW} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2w_{11,21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2w_{12,22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4w_{1q1,2q1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4w_{1q2,2q2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4w_{1q3,2q3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4w_{1q4,2q4} \end{pmatrix} \tag{20}$$

$$w_{11,21} = \frac{n_{11}+n_{21}}{n_1+n_2}, \quad w_{12,22} = \frac{n_{12}+n_{22}}{n_1+n_2},$$
$$w_{1q1,2q1} = \frac{n_{1q1}+n_{2q1}}{n_1+n_2}, \quad w_{1q2,2q2} = \frac{n_{1q2}+n_{2q2}}{n_1+n_2},$$
$$w_{1q3,2q3} = \frac{n_{1q3}+n_{2q3}}{n_1+n_2}, \quad w_{1q4,2q4} = \frac{n_{1q4}+n_{2q4}}{n_1+n_2}.$$

Thus, the lower the number of transacted houses in a given period (quarter, month), the noisier the computed index and thus the larger the variance. Note that the trace of the weights matrix W_{TW} is equal to the number of layers in the hierarchy as $w_{11,21} + w_{12,22} = 1$ and similarly $\sum_{i=1}^{4} w_{1qi,2qi} = 1$.

To illustrate the solution to the reconciled price indices we consider the simple two-layer hierarchy of annual and semi-annual that was presented in Section 2.1. In this case the projection matrix is as follows:

$S(S'WS)^{-1}S'W =$

$$\frac{1}{(1+2w_{11,21})(1+2w_{12,22})-1} \begin{pmatrix} 2 & 4w_{11,21}w_{12,22} & 4w_{11,21}w_{12,22} \\ 2w_{12,22} & 2w_{11,21}(2w_{12,22}+1) & -2w_{12,22} \\ 2w_{11,21} & -2w_{11,21} & 2w_{12,22}(2w_{11,21}+1) \end{pmatrix}, \tag{21}$$

The solution for the reconciled price indices is as follows:

$$\ln \hat{P}_{1,2} = \frac{2\ln P_{1,2} + 2w_{11,21}w_{12,22}\ln P_{11,21} + 2w_{11,21}w_{12,22}\ln P_{12,22}}{(1+2w_{11,21})(1+2w_{12,22})-1} \tag{22}$$

$$\ln \hat{P}_{11,21} = \frac{4w_{12,22} \ln P_{1,2} + 2w_{11,21}(2w_{12,22}+1) \ln P_{11,21} - 2w_{12,22} \ln P_{12,22}}{(1+2w_{11,21})(1+2w_{12,22})-1}, \quad (23)$$

$$\ln \hat{P}_{12,22} = \frac{4w_{11,21} \ln P_{1,2} + 2w_{12,22}(2w_{11,21}+1) \ln P_{12,22} - 2w_{11,21} \ln P_{11,21}}{(1+2w_{11,21})(1+2w_{12,22})-1}. \quad (24)$$

5.2 The Time Series Dimension of the Reconciliation

Up to now the discussion has referred to y, the stacked annualised price changes of all frequencies in the hierarchy at a given point in time. To implement the reconciliation, either least squares, (3) or weighted least squares, (19) can be run at each time period[3] to obtain a reconciled set of annualised price changes $\hat{y}$. That is, an estimate of a vector of parameters, β, is obtained for each time period by running model (25) T times,

$$y_t = S\beta_t + \epsilon_t; \qquad t = 1, \ldots, T \quad (25)$$

We can now ask whether estimating β_t independently for each available time period is the most reasonable econometric approach. An alternative is given by estimating (25) as a time-varying parameter model maintaining the structure and assumptions of the reconciliation. In the application we construct indices from three sets of estimates: period-by-period OLS and WLS, and from following time-varying parameter model:

- Maintain the WLS assumption: $\epsilon_t \sim N(0, H_t)$, where $H_t = (W_{TW_t})^{-1}$
- Assume $\beta_t = \beta_{t-1} + \eta_t$ and the covariance of η_t is $Q = diag((S'S)^{-1})$ which can be easily verified to be $Q = \sigma_\eta^2 I$, where σ_η^2 is a constant and I is an identity matrix.
- Assume at $t = 0$, the covariance of β_t is Q

Under the above assumptions we can derive the Kalman filter estimate of $S\beta_t$.[4]

$$\hat{y}^{KF} = S\tilde{\beta}_t^{TV} = (I - G_t)S\tilde{\beta}_{t-1}^{TV} + G_t y_t \quad (26)$$

where, $G_t = S'(SS' + (\frac{1}{t\sigma_\eta^2})H_t)^{-1}$ and $(\frac{1}{t\sigma_\eta^2})H_t$ is the inverse signal-to-noise ratio in the state-space system.[5]

By computing $S\beta_t$ using equation (26) we expect the reconciled price indices to be smoother. This will have consequences for the reconciled period-on-period indices

[3] Real time estimation is easily achieved by constructing the annualised stacked vector of price changes when a new observation becomes available for the highest frequency in the set, e.g. monthly. We implement real time estimation for a three layer hierarchy in Section 8.

[4] The state-space system is given by equation (25) and $\beta_t = \beta_{t-1} + \eta_t$. To derive the expression we write the system in the innovation form of the Kalman filter, $S\beta_{t+1|t} = S\beta_{t|t-1} + G_t\nu_t$, where G_t is the Kalman gain and ν_t is the one-step ahead prediction error.

[5] If we were to assume spherical errors, $H = \sigma_\epsilon^2 I$, the signal-to-noise ratio would be given by $\frac{t\sigma_\eta^2}{\sigma_\epsilon^2}$

(e.g., quarterly, monthly) that can be derived from the reconciled vectors of annualised indices.

In Section 8 we compare reconciled indices for property prices at the annual, quarterly and monthly frequencies.

6 Backing out Higher Frequency Indices

Our method as described so far generates reconciled annualised indices. The starting point of these indices can be any month or quarter in the year, For example, we have annual indices beginning in each quarter, denoted as follows: $p^R_{1q1,2q1}, p^R_{1q2,2q2}, p^R_{1q3,2q3}$, and $p^R_{1q4,2q4}$.

From a policy perspective reconciled quarterly and monthly indices would also be useful. We show below how reconciled quarterly indices ($p^R_{1q1,1q2}$, $p^R_{1q2,1q3}$, $p^R_{1q3,1q4}$) can be derived from reconciled annualised indices. An analogous approach could be used to derive reconciled monthly indices.

Recursive algorithms

We can back out quarterly indices from the annualized indices, by exploiting the following relationships:

$$p^R_{1q1,2q1} + p^R_{2q1,2q2} = p^R_{1q1,1q2} + p^R_{1q2,2q2}. \tag{27}$$

This can be rearranged as follows:

$$p^R_{2q1,2q2} = p^R_{1q1,1q2} + p^R_{1q2,2q2} - p^R_{1q1,2q1}. \tag{28}$$

Similarly for the next three quarters we have that:

$$p^R_{2q2,2q3} = p^R_{1q2,1q3} + p^R_{1q3,2q3} - p^R_{1q2,2q2} \tag{29}$$
$$p^R_{2q3,2q4} = p^R_{1q3,1q4} + p^R_{1q4,2q4} - p^R_{1q3,2q3} \tag{30}$$
$$p^R_{2q4,3q1} = p^R_{1q4,2q1} + p^R_{2q1,3q1} - p^R_{1q4,2q4}. \tag{31}$$

Hence once we have reconciled quarterly indices for the first year (i.e., $p^R_{1q1,1q2}$, $p^R_{1q2,1q3}$, $p^R_{1q3,1q4}$, and $p^R_{1q4,2q1}$), we can use these formulas to compute reconciled quarterly indices for the second year, and likewise for the year after that, etc. A simple way of starting this recursive algorithm is to set the reconciled quarterly indices equal to the unreconciled quarterly indices for the first year:

$$p^R_{1q1,1q2} = p_{1q1,1q2}, \quad p^R_{1q2,1q3} = p_{1q2,1q3}, etc. \tag{32}$$

However, the recursive formulations of equations (28)-(31) induce a spurious memory of lagged terms. To see this consider the reconciled price change from Q1 to Q2 for year

3,

$$p^R_{3q1,3q2} = p^R_{2q1,2q2} + p^R_{2q2,3q2} - p^R_{2q1,3q1} \tag{33}$$

and replace $p^R_{2q1,2q2}$ by equation (28). We get

$$p^R_{3q1,3q2} = [p^R_{1q1,1q2} + p^R_{1q2,2q2} - p^R_{1q1,2q1}] + p^R_{2q2,3q2} - p^R_{2q1,3q1} \tag{34}$$

One alternative is the following:

$$p^R_{3q1,3q2} = p_{2q1,2q2} + p^R_{2q2,3q2} - p^R_{2q1,3q1} \tag{35}$$

Replacing $p^R_{2q1,2q2}$ with $p_{2q1,2q2}$ in (35) avoids the spurious memory problem at the expense of using an unreconciled index. Another alternative is to average the unreconciled and reconciled indices as follows:

$$p^R_{3q1,3q2} = [(p_{2q1,2q2} + p^R_{2q1,2q2})/2] + p^R_{2q2,3q2} - p^R_{2q1,3q1} \tag{36}$$

This averaged formula in (36) will still contain some spurious memory, although it will be more dampened than in (33).

Another alternative is to use a system of equations approach.

A system of equations approach

Each equation below relates a reconciled annual index to a corresponding chain of reconciled quarterly indices. For example, for k years we have

$$p^R_{1q1,1q2} + p^R_{1q2,1q3} + p^R_{1q3,1q4} + p^R_{1q4,2q1} = p^R_{1q1,2q1} \tag{37}$$

$$p^R_{1q2,1q3} + p^R_{1q3,1q4} + p^R_{1q4,2q1} + p^R_{2q1,2q2} = p^R_{1q2,2q2} \tag{38}$$

$$\vdots \tag{39}$$

$$p^R_{(k-1)q4,kq1} + p^R_{kq1,kq2} + p^R_{kq2,kq3} + p^R_{kq3,kq4} = p^R_{(k-1)q4,kq4} \tag{40}$$

Assume that we have n_q quarters in our records which correspond to $n_q - 4$ reconciled annual indices (the right hand side of equations (37)-(40)).[6] However, we need to recover $n_q - 1$ reconciled quarterly indices (the elements of the left hand side of equations (37)-(40)), i.e., the equation system is not identified. To overcome this issue, we add three further equations where the reconciled quarterly index is set to equal its unreconciled counterpart. For example,

$$p^R_{1q1,1q2} = p_{1q1,1q2}, \;\; p^R_{1q2,1q3} = p_{1q2,1q3}, \;\; p^R_{1q3,1q4} = p_{1q3,1q4}. \tag{41}$$

Now, the equation system (37)-(41) has a unique solution.[7] However, this outcome depends on the choice made in equation (41). To make the final reconciled quarterly

[6]Note that this method works for any number n_q. It is not necessary to consider only 'full' years.

[7]This method builds on (35) in that it again uses some unreconciled indices, although only a small number of them.

index independent of this choice, we propose to consider all possible combinations and to average with equal weight over all produced indices in a GEKS-type procedure. As the number of combinations $\binom{n_q-1}{3}$ rises with the number of quarters in the sample and gets quickly infeasible, we consider only the case of consecutive quarterly indexes. This has the advantage of preserving the short-term trend in the reconciled period-on-period indices which is not necessarily the case in the recursive algorithm.

Non-revisability

A disadvantage of the system of equations approach introduced is that the whole history of reconciled quarterly indices gets revised when a new quarter is added to the analysis. To prevent backward revisions, we need to modify slightly the backing-out algorithm for our real-time procedure and propose (i) a rolling window version and (ii) a combination of the system of equations and the recursive algorithm. Note that the recursive algorithm produces non-revisable real-time indices by construction.

For the rolling window version, we fix the number of quarters to be considered, say by $n_q^0 < n_q$, when applying the system of equations approach. This gives the first $n_q^0 - 1$ reconciled period-on-period indices. With each new quarter, the window of fixed n_q^0 quarters is rolled forward. The system of equations approach is applied and the newest reconciled quarterly index is spliced to the existing series. Different variants of splicing techniques are discussed, for example, in De Haan et al. (2020) and Hill et al. (2020b). Instead of rolling the whole system forward, when quarter $n_q^0 + 1$ is added, one also could carry forward with one of the variants of the recursive algorithm.

We denote the method of equations by **full system**, its rolling window version by **RW system**. With the RW method, the index provider can choose the window length or metric to select an optimal window (discussed in Section 7). One more alternative that we consider here is to use the full system method for a sample of periods at the beginning of the sample and then switch to the average recursive method described in (36) thereafter. Again the index provider can decide the length of the initial window over which the full system method is used. We refer to this method as **system recursive**.

7 Measuring the Quality of an Index

To verify that our reconciled indices are more reliable that their unreconciled counterparts, we use a variant on the quality measure proposed by Hill et al. (2020a) based on repeat-sales. Suppose a property i sells in periods t and $t + k$. For this repeat sale we can compare the actual observed price change $p_{i,t+k}/p_{i,t}$ with the corresponding price change obtained from an index, denoted by P_{t+k}/P_t.

We now calculate the difference of the logs of these price relatives for each property

i as follows:

$$d_i = \ln\left(\frac{P_{t+k}}{P_t}\right) - \ln\left(\frac{p_{i,t+k}}{p_{i,t}}\right). \tag{42}$$

Averaging over all repeat-sales properties i, with N_{RS} denoting the total number of repeat-sales in our records, our measure of index quality is given by the mean-squared difference of the d_i terms:

$$IQ = \frac{1}{N_{RS}} \sum_{i=1}^{N_{RS}} (d_i)^2 \tag{43}$$

Taking logs before computing the mean-squared difference ensures that price increases and price declines are treated symmetrically. The index with the smallest IQ is preferred because it best approximates the price change over time on average. This presupposes that the repeat-sales sample is representative for the housing market of interest (i.e., lemons bias and other sampling problems should be of limited concern).[8]

8 Application to Sydney Eastern Suburbs with Real Time Reconciliation

In this section we present an example of a reconciliation of price indices for residential housing. We consider here a hierarchy of annual, quarterly and monthly imputed indices which have been computed using the standard hedonic imputation approach from the price index literature with the transactions level data.

It is easy to verify that for an annual, quarterly and monthly system, S and y are defined as follows in this three-level hierarchy:

[8]Lemons bias refers to a situation where lower quality starter properties sell more frequently that other properties and hence are over-represented in the repeat-sales sample. A bias arises if these starter properties follow a different price trend from the rest of the market (see, Clapp and Giaccotto, 1992, Gatzlaff and Haurin, 1997, and Shimizu, Nishimura and Watanabe, 2010). A simple check would be the comparison of the price distributions of repeat-sales and single-sales observations. For our sample of Eastern suburbs of Sydney, we do not find significant differences (see Figure 1). An alternative would be the use of the whole sample (i.e., not only repeat-sales) together with a model based approach. Then imputed prices would be needed to replace one or both of the observed prices in the price relative $p_{i,t+k}/p_{i,t}$.

$$
S = \begin{bmatrix} \mathbf{j}'_{24} \\ \mathbf{I}_4 \otimes \mathbf{j}'_6 \\ \mathbf{I}_{12} \otimes \mathbf{j}'_2 \end{bmatrix} \quad y = \begin{pmatrix} \ln P_{1,2} \\ (1/4)\ln P_{1q1,2q1} \\ \vdots \\ (1/4)\ln P_{1q4,2q4} \\ (1/12)\ln P_{1m1,2m1} \\ (1/12)\ln P_{1m2,2m2} \\ \vdots \\ (1/12)\ln P_{1m11,2m11} \\ (1/12)\ln P_{1m12,2m12} \end{pmatrix}.
$$

where,

$\mathbf{j}'_m$ is an m row vector of 1's

$\mathbf{I}_m$ is an identity of size m

and $\otimes$ is the Kronecker tensor product[9]

The unreconciled indices are computed from models using with data from the Eastern Suburbs of Sydney, Australia covering the period 2001-2014 at each of the three frequencies, annual, quarterly and monthly. The Eastern Suburbs of Sydney contain 14 postcodes, and some of the most famous area of the city. Table 1 lists the suburb and postcodes used to compute the basic indices.

The reconciliation is conducted in *real time*. That is every month the vector of stacked indices, y, can be formed and reconciled. For example, in year 1, month 2 we compute an annual index $P_{1m2,2m2}$, a quarterly index $P_{1m2,1m5}$, and a monthly index $P_{1m2,1m3}$. As output, every month we obtain reconciled annual, quarterly and monthly indices ($P^R_{1m2,2m2}$, $P^R_{1m2,1m5}$, and $P^R_{1m2,1m3}$).

In our records, we have 3 797 repeat-sales out of a total of 23 454 family home transactions. Table 1 lists the suburbs with the corresponding postcodes and number of observations. Figure 1 compares the distributions of log-prices for the repeat-sales sample and the sample of houses that transact only once. The two distributions overlap well and this ensures that any sampling issues, such as lemons bias, are not problematic in the computation of the index quality measure (IQ) in our application.

Running our real-time algorithm gives us reconciled annual, quarterly and monthly indices every month.[10] The left side of Figure 2 shows those annualized indices. We use the following notation: **U** denotes an index that has not been reconciled,

R(OLS) is the index that has been reconciled using the Least Squares projection matrix, **R(WLS)** denotes the reconciled indices obtained using Weighted-Least

[9]The Kronecker tensor product of matrices A and B is an $m \times p - by - n \times q$ matrix formed by taking all possible products between the elements of A (an $m - by - n$ matrix) and B (a $p - by - q$ matrix)

[10]For ease of interpetation, the results presented here are based on the usual calender representation of quarterly and annual indices, that is, Q1 covers January, February, and March, Q2 covers April, May and June, etc.

Table 1: Sydney Eastern Suburbs, Period: 2001–2014

Suburb	Postcode	total transactions	repeat-sales
Paddington	2021	2535	486
Bondi Junction	2022	1499	281
Bellevue Hill	2023	1018	153
Waverley	2024	1241	203
Woollahra	2025	1203	224
Bondi	2026	2287	393
Edgecliff	2027	350	51
Double Bay	2028	350	61
Rose Bay	2029	760	117
Vaucluse	2030	1963	270
Randwick	2031	2527	432
Kingsford	2032	1044	151
Kensington	2033	627	78
Coogee	2034	1301	192
Pagewood	2035	2784	442
Matraville	2036	1965	263

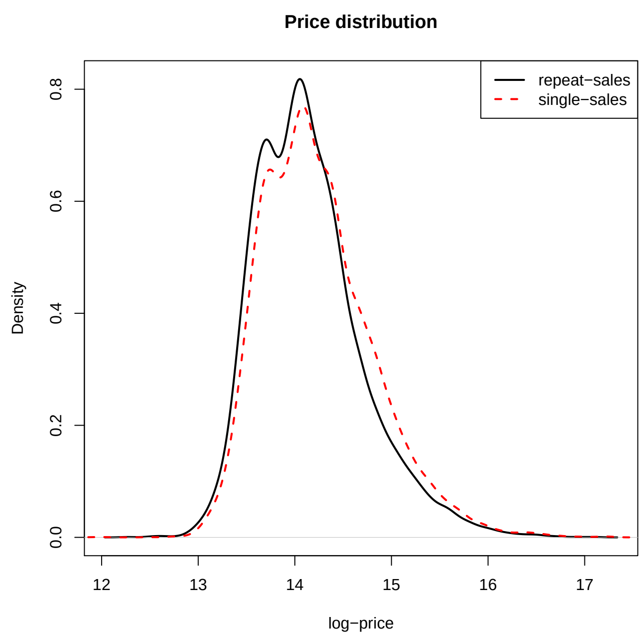

Figure 1: Log-price distribution of single- vs. repeat-sales

Table 2: Measuring the quality of the reconciled indices

Frequency	Method	U	R(OLS)	R(KF)	R(WLS)
Annual		0.042507			
			0.042601	**0.042237**	0.042556
Quarterly		0.041189			
	pure recursive		0.104763	0.101598	0.104730
	no memory		0.041599	**0.041077**	0.041582
	averaged		0.041675	**0.040693**	0.041682
	full system		0.042520	0.041801	0.042374
	system recursive		**0.040571** (40)	**0.040339** (40)	**0.040541** (40)
	RW system		0.041277 (15)	**0.041095** (16)	**0.041182** (15)
Monthly		0.043518			
	pure recursive		0.111889	0.108850	0.111946
	no memory		0.045119	0.044120	0.044887
	averaged		0.045262	**0.043217**	0.044803
	full system		**0.043378**	**0.041794**	**0.042530**
	system recursive		**0.040727** (46)	**0.040723** (47)	**0.040516** (46)
	RW system		**0.043271** (62)	**0.041588** (49)	**0.042349** (61)

Squares projection (from (19)), and **R(KF)** is the reconciled indices obtained using the Kalman Filter predictor (from (26))[11]. Table 2 presents the measured index quality (IQ), equation (43), for the different methods discussed in Section 6 to back-out the higher frequency indices (in this case the quarterly and monthly frequencies). For the hybrid methods a window length is required. We choose the window length that minimises the IQ measure in each case, and the number of periods ($n_f^0, f \in \{q, m\}$) are reported in brackets. For the annual frequency no backing-out algorithm is necessary and the IQ measure (43) can be applied directly. The methods which yield an IQ measure lower than that of the unreconciled indices are highlighted in bold.

A visual comparison of the alternative reconciliation methods is presented in Figure 2, while a comparison of alternative backing out methods is presented in Figure 3.

The left side of Figure 2 shows the annualised reconciled indices across all frequencies. The right side shows the chained period-on-period indices for the annual, quarterly, and monthly frequency using the method **system recursive** for backing out the period-on-period indices. Note that for the monthly frequency, the reconciled indices show a much larger price increase over the whole sample period than that shown by the

[11] We assume heteroscedastic variance for the measurement equation error, $H_t = (W_{TW_t})^{-1}$

unreconciled index.

Figure 3 shows a comparison of the backed out quarterly and monthly indices from the R(KF) reconciled annualised indices. All alternative backing out approaches, recursive, system and hybrid, are shown. In this figure it is clear that the purely recursive approach is not correctly capturing the trend in price changes, which demonstrates the problem with the spurious memory lags indicated in Section 6. For the monthly frequency, the recursive (no memory and averaged) give very similar indices, while the system based alternatives seem to back out indices that are closer to each other for most of the sample with some divergence after 2013 when the system recursive shows higher price changes than any of the other indices.

Summarising the findings we note:

(i) the hybrid method **system recursive** consistently outperforms all other backing out algorithms.

(ii) the **pure recursive** method, prone to error accumulation due to its spurious memory of lagged terms, gives the worst results for all frequencies in terms of index quality.

(iii) the best reconciled indices are those obtained using the time-varying parameter, R(KF) predictor. For the monthly frequency, the IQ measure is close but slightly lower for R(WLS) than that from R(KF), indicating that it might recover very similar indices.

Based on these findings, we recommend the use of the R(KF) in combination with the **system recursive** method for backing-out period-on-period indices for all frequencies.

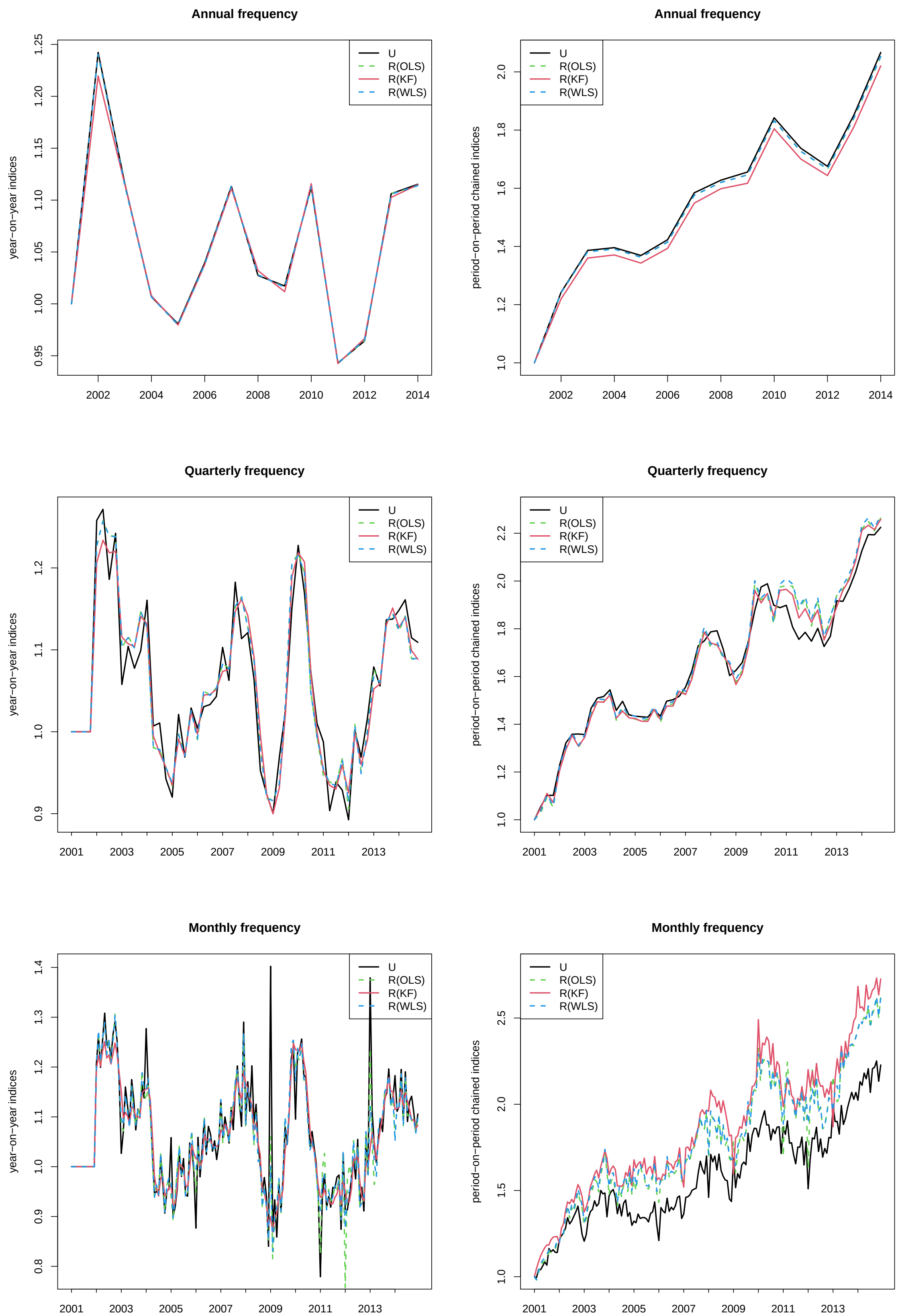

Figure 2: Left: year-on-year indices, right: chained period-on-period indices. **U** unreconciled index, **R(OLS)** Least Squares predictor (eq. (3)), **R(KF)** Kalman Filter predictor (eq. (26)), **R(WLS)** Weighted Least Squares predictor (eq. (19)).

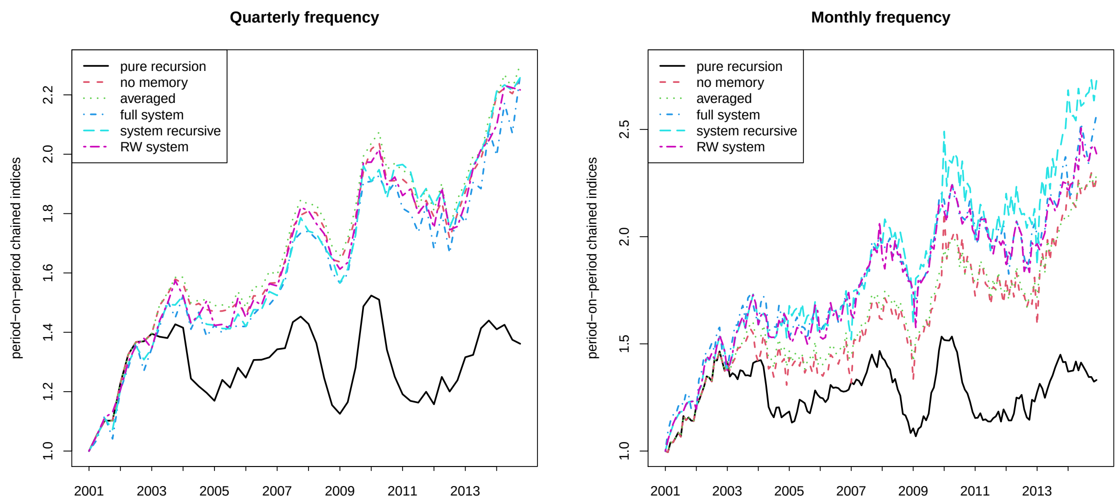

Figure 3: Backed Out Quarterly and Monthly Indices Using Kalman Filter Predictor Reconciliation (second column of Table 2)

9 Conclusion

The inconsistencies between indices computed at difference frequencies provide an opportunity to improve their reliability. The reconciliation method we have proposed here exploits this insight. We set the problem by writing the hierarchy across temporal dimensions as a stacked vector, and show how the reconciliation is the solution to a constrained least squares problem.

We have considered a number of variants on the basic method. Our variants differ along two dimensions. We propose three ways of reconciling at the lowest frequency (here annual). These are based on using three alternative estimators of the reconciliation model, which are a Least Squares predictor, R(OLS), a Weighted Least Squares R(WLS) predictor, and a Kalman filter, R(KF) predictor. The reconciliation is run in real time and thus produces new annualised reconciled annual, quarterly and monthly indices *with every month of new data*. We then considered six ways for backing out reconciled higher frequency indices, that is the reconciled period-on-period price changes. Each of the alternative backing out methods can be combined with any of the annualised reconciled indices obtained from R(OLS), R(WLS) or R(KF).

We developed a measure of index quality to allow us to choose the best performing combination of reconciliation and backing out method.

In our empirical application to transaction level data from the Eastern Suburbs of Sydney, Australia, the best performing combination is when using R(KF) with a system recursive backing out approach. We find a rolling window backing out approach also performs well. In both cases, the reconciled indices are of higher quality (more accurate) than the unreconciled indices at the higher frequencies of quarterly and monthly.

In conclusion, our approach produces improved indices at all frequencies. Furthermore, our method provides more timely indices. For example, rather than having to wait until the end of the year to obtain a new annual index, or the end of a quarter for a new quarterly index, our method produces a new annual and quarterly index every month. These innovations – greater reliability and timeliness – can help users such as central banks and governments make more informed decisions.

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