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SEASONAL COMMODITIES, HIGH INFLATION AND INDEX NUMBER THEORY

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Abstract

This paper studies the problems of measuring economic growth under conditions of high inflation. Traditional bilateral index number theory implicitly assumes that variations in the price of a commodity within a period can be ignored. In order to justify this assumption under conditions of high inflation, the accounting period must be shortened to a quarter, month or possibly a week. However, once the accounting period is less than a year, the problem of seasonal commodities is encountered; i.e., in some subannual periods, many seasonal commodities will be unavailable, and hence the usual bilateral index number theory cannot be applied. The present paper systematically reviews the problems of index number construction when there are seasonal commodities and high inflation. Various index number formulae are justified from the viewpoint of the economic approach to index number theory by making separability assumptions on consumers' intertemporal preferences. We find that accurate economic measurement under conditions of high inflation is very complex. Statistical Agencies should produce at least three different types of index: (i) year over year "monthly" price and quantity indexes; (ii) a short term "month to month" price index of nonseasonal commodities and (iii) annual Mudgett-Stone quantity indexes that use the short term price index in (ii) to deflate the seasonal prices. In section 8, it is shown how the annual Mudgett-Stone quantity indexes can be calculated for moving years as well as for calendar years. These moving year indexes can be centered and the centered indexes can serve as "monthly" seasonally adjusted indexes at annual rates. In section 9, this index number method of seasonal adjustment is compared with traditional time series methods of seasonal adjustment. The paper is also related to the accounting literature on adjusting for changes in the general price level.

Key Words: Aggregation of commodities, consumer theory, index numbers, inflation, seasonal adjustment, separability, time series.

JEL Classification Numbers: B23, C43, D11, D91, E31, M4

1. Introduction

Ever since the German hyperinflation of the twenties, accountants[1] have noted that high inflation causes historical cost accounting measures of income and wealth to become virtually useless. One way to restore credibility to business accounts would be is to deflate current values by appropriate price indexes. However, the construction of price indexes is not straightforward under conditions of high inflation–particularly when seasonal commodities are present. Recently, Hill (1995) addressed some of these problems in the context of adapting the United Nations (1993) system of national accounts to high inflation situations. This paper can be regarded as an extension of Hill's contributions, taking account of seasonal commodities.

Before describing the contents of the paper, we address some preliminary questions.

What are seasonal commodities? They are commodities which are either (i) not available during certain seasonsor (ii) are always available but there are fluctuations in prices or quantities that are synchronized with the season or time of year.[2]

What are the sources of seasonal fluctuations in prices or quantities? There are two main sources:[3] (i) climate and (ii) custom. In the first category, fluctuations in temperature, precipitation, wind and hours of daylight cause fluctuations in demand for things such as ice skates, fuel oil, umbrellas, snow tires, seasonal clothing, and electricity. In more generic terms, climatic changes cause fluctuations in energy demands, recreational activities, and food consumption patterns. In fact, seasonal fluctuations are present in almost all sectors of most economies.

What are the implications of seasonality for index number theory? If we break the year up into M seasons (e.g., $M = 4$ if the season is a quarter or $M = 12$ if the season is a month, then the existence of type (i) seasonal commodities in the set of goods we are aggregating over means the dimension of the commodity space will not be constant. Thus it will be impossible[4] to apply the usual bilateral index number theory.

Even if all commodities were available in all seasons, the existence of type (ii) seasonal commodities may mean that bilateral indexes that are exact for an underlying utility function cannot be justified[5]. The economic approach assumes that the seasonal aggregator function is the same in each season being compared, which is not a reasonable assumption if climate and customs interact with tastes. This suggests that type (ii) seasonal commodities should be further classified into sub types (a) and (b).

A type (ii) seasonal commodity is defined to be of sub type (a) if its seasonal quantity fluctuations can be rationalized by utility maximizing behavior over a set of seasons where the prices fluctuate but the utility aggregator function remains unchanged, and of type (ii) (b) if its quantity fluctuations cannot be rationalized by maximizing an unchanging utility function over the periods in question.[6] An example may be helpful. As harvest conditions vary, the price of potatoes in my local supermarket varies and I purchase more potatoes as the price falls and less as it rises. On the other hand, the price of beer remains quite constant throughout the year but my consumption greatly during the summer. Weather shifts my seasonal demand functions for beer which is a type (ii) (b) seasonal commodity and but not potatoes which are a type (ii) (a) seasonal commodity. The usual economic approach to index number theory can be applied to type (ii) (a) seasonal commodities but not to type (ii) (b) ones.

The problem of index number construction when there are seasonal commodities has a long history; e.g., see Flux (1921; 184-185), Crump (1924; 185), Bean and Stine (1924), Mudgett (1955), Stone (1956), Rothwell (1958), Zarnowitz (1961), Turvey (1979) and Balk (1980a), (1980b) (1980c) (1981). However, what has been missing is an exposition of the assumptions on the consumer's utility function[7] that are required to justify a particular formula. In the present paper, we will systematically list separability assumptions[8] on intertemporal preferences that can be used to justify various seasonal index number formulae from the viewpoint of the economic approach to index number theory.[9]

We now set out the general model of consumer behavior that we will specialize in subsequent sections. Suppose that there are M seasons in the year and the Statistical Agency has collected price and quantity data on the consumer's purchases for $1+T$ years.[10] Suppose further that the dimension of the commodity space in each season remains constant over the $T+1$ years; i.e., season m has N_m commodities for $m = 1, \ldots, M$. For season m of year

t, we denote the vector of positive prices facing the consumer by $p^{tm} \equiv [p_1^{tm}, p_2^{tm}, \ldots, p_{N_m}^{tm}]$ and the vector of commodities consumed in by $q^{tm} \equiv [q_1^{tm}, q_2^{tm}, \ldots, q_{N_m}^{tm}]$. It will prove convenient to have notation for the annual price and quantity vectors, so we define these by:

$$p^t \equiv [p^{t1}, p^{t2}, \ldots, p^{tM}]; q^t \equiv [q^{t1}, q^{t2}, \ldots, q^{tM}]; \quad t = 0, 1, \ldots, T. \tag{1}$$

To apply the economic approach to index number theory, it is necessary to assume that the observed quantities of q_n^{tm} are a solution to an optimization problem involving the observed prices p_n^{tm}. We follow Fisher (1930), Hicks (1946; 121-126) and Pollak (1989; 72) and assume that the intertemporal quantity vector $[q^0, q^1, \ldots, q^T]$ is a solution to the following intertemporal utility maximization problem:

$$max_{x^0,x^1,\ldots,x^T}\{U(x^0, x^1, \ldots, x^T) : \Sigma_{t=0}^T \delta_t p^t \cdot x^t = W\} \tag{2}$$

where $x^t \equiv [x^{t1}, x^{t2}, \ldots, x^{tm}]$ and each seasonal quantity vector x^{tm} has the dimensionality of $q^{tm}, p^t \cdot x^t \equiv \Sigma_{m=1}^M p^{tm} \cdot x^{tm}$ and $p^{tm} \cdot x^{tm} \equiv \Sigma_{n=1}^{N_m} p_n^{tm} x_n^{tm}, U$ is the consumer's intertemporal preference function (assumed to be continuous and increasing), $\delta_t > 0$ is an annual discount factor and "wealth" W is the consumer's current and expected future discounted "income" viewed from the perspective of the beginning of year 0. If the consumer can borrow and lend at a constant annual nominal interest rate r, then $\delta_0 \equiv 1$ and

$$\delta_t = 1/(1 + r)^t, \quad t = 1, 2, \ldots, T. \tag{3}$$

Since we are assuming that the quantity vector $[q^0, q^1, \ldots, q^T]$ is a solution to (2), it must satisfy the intertemporal budget constraint in (2) so we can replace W by

$$W \equiv \Sigma_{t=0}^{T} \delta_t p^t \cdot q^t. \tag{4}$$

Our assumptions are admittedly unrealistic. The consumer is assumed to: (i) know future spot prices p^t; (ii) know his or her future income streams; (iii) be able to freely borrow and lend between years at the same rates and (iv) have unchanging tastes over years. Under these assumptions, the consumer at the beginning of year 0 chooses a sequence of annual consumption plans, $q^t, t = 0, 1, \ldots, T$, and sticks to them.

The economic approach to index numbers requies strong assumptions. Some advantages of this approach are: (i) it allows for substitution in response to changes in the prices; (ii) it provides a concrete framework which can be used to assess operational alternatives that occur when a Statistical Agency constructs an index number[11] and (iii) it leads to definite recommendations about the choice of functional forms for index number formulae which can then be evaluated from other perspectives, such as the test approach.

Having made our basic economic assumptions (namely that the observed sequence of annual quantity vectors $[q^0, q^1, \ldots, q^t]$ solves (2) with W defined by (4)), now make additional assumptions on the structure of the intertemporal utility function U.

In section 2, we show how (2) can be specialized to yield the annual indexes first proposed by Mudgett (1955; 97) and Stone (1956; 74-75). In section 3, we note that our Hicksian intertemporal utility maximization problem (2) needs to be modified when inflation is high. The annual discount factors δ_t that appear in (2) and (4) do not provide an adequate approximation to the consumer's intertemporal problem with even moderate inflation between seasons: we need to introduce between season intra year discount rates as well. In section 4, we show that when there are seasonal commodities, the use of annual sums of seasonal quantities and the corresponding annual unit values are unsatisfactory as annual quantity and price aggregates.

In section 5, we leave the problems involved in the construction of annual aggregates and turn our attention to the construction of seasonal aggregates. In this section, we consider the construction of year over year seasonal aggregates. In section 6, we examine the consistency of the year over year seasonal aggregates of section 5 with the annual indexes of sections 2 and 3. In section 7, we get into the heart of the seasonal aggregation problem and consider methods for obtaining valid season to season measures of price change when there are seasonal commodities. In section 8, we consider how to extend the scope of the annual calendar year indexes of section 3 to "moving" year comparisons. In section 9, we indicate how the moving year indexes of section 8 can be centered. These centered indexes provide an index number solution to the problem of seasonal adjustment. Section 10 concludes.

2. The Construction of Annual Indexes Under Conditions of Low Inflation

In the Mudgett (1955; 97)-Stone(1956; 74-75)[12] approach to annual index numbers when there are seasonal commodities, we need to restrict the consumer's intertemporal utility function U as follows: there exist F and f such that

$$U(x^0, x^1, \ldots, x^T) = F[f(x^0), f(x^1), \ldots, f(x^T)] \tag{5}$$

where f is a linearly homogeneous, increasing and concave annual utility function[13] and F is an intertemporal utility function that is increasing and continuous in its $T+1$ annual utility arguments. The annual utility function f is assumed to be unchanging over time.

If $q^0, q^1, \ldots, q^T$ solves (2) with W defined by (4) and U defined by (5), then it can be seen that q^t, the observed annual consumption vector for year t, is a solution to the following year t utility maximization problem:

$$max_{x^t}\{f(x^t) : p^t \cdot x^t = p^t \cdot q^t\} = f(q^t); \quad t = 0, 1, \ldots, T. \tag{6}$$

Now we are in a position to apply the theory of exact index numbers.[14] Assume that the bilateral quantity index $Q(p^s, p^t, q^s, q^t)$ is exact for the linearly homogeneous aggregator function f. Then we have

$$f(q^t)/f(q^s) = Q(p^s, p^t, q^s, q^t); \quad 0 \le s, t \le T. \tag{7}$$

As an example of (7), suppose that the annual aggregator function f is $f(x) \equiv (x \cdot Ax)^{1/2}$ where A is a symmetric N^* by N^* matrix of constants satisfying certain regularity conditions. This functional form is flexible; i.e., it can provide a second order approximation to an arbitrary differentiable linearly homogeneous function. The quantity index that is exact for this functional form is the Fisher (1922) ideal quantity index Q_F[15]

$$Q_F(p^s, p^t, q^s, q^t) \equiv [p^t \cdot q^t p^s \cdot q^t / p^t \cdot q^s p^s \cdot q^s]^{1/2}. \tag{8}$$

Since Q_F is exact for a flexible functional form, it is a superlative index.[16]

Given any bilateral quantity index Q, its associated price index P can be defined as follows using Fisher's (1911; 403) weak factor reversal test[17]:

$$P(p^s, p^t, q^s, q^t) \equiv p^t \cdot q^t / p^s \cdot q^s Q(p^s, p^t, q^s, q^t). \tag{9}$$

Given any linearly homogeneous, increasing and concave aggregator function f, its dual unit cost function can be defined for strictly positive prices $p >> 0_{N^*}$ as:

$$c(p) \equiv min_x\{p \cdot x : f(x) = 1\}. \tag{10}$$

When the utility function f is linearly homogeneous, the Konüs (1924) price index between periods s and t reduces to the ratio of the unit cost functions evaluated at the period s and t prices, $c(p^t)/c(p^s)$. If the bilateral quantity index Q is exact for f, then its companion bilateral price index P defined by (9) is exact for the unit cost function c dual to f; i.e., in addition to (7), we also have

$$c(p^t)/c(p^s) = P(p^s, p^t, q^s, q^t); 0 \leq s, t \leq T. \tag{11}$$

As an example of (11), suppose that the annual aggregator function is the homogeneous quadratic aggregator $f(x) \equiv (x \cdot Ax)^{1/2}$ and that c is its unit cost dual function. Then (11) holds with $P = P_F$ where the Fisher ideal price index P_F is defined by

$$P_F(p^s, p^t, q^s, q^t) \equiv [p^t \cdot q^t p^t \cdot q^s / p^s \cdot q^t p^s \cdot q^s]^{1/2}. \tag{12}$$

The above analysis seems to indicate that the construction of annual price and quantity indexes when there are seasonal commodities is straightforward: simply regard each "physical" commodity in each season as a separate economic commodity and apply ordinary index number theory to the enlarged annual commodity space. However, this does not work when there is sever or even moderate inflation between seasons within the year.

3. The Construction of Annual Indexes Under Conditions of High Inflation

In the previous section, a discount rate δ_t was used to make the prices in year t comparable to the base year prices. With low inflation, this is an acceptable approximation to the consumer's intertemporal choice problem. However, when inflation is high, we can no longer neglect interseasonal interest rates.

Consider the budget constraint in (2). We now interpret δ_t as the discount factor that makes one dollar at the beginning of year t equivalent to one dollar at the beginning of year 0. From the beginning of t to the middle of season m in t, another discount factor is required, say ρ_{tm}, which will make a dollar at the beginning of t equivalent to a dollar in the middle of season m of t. Thus the budget constraint in (2) must be replaced by an intertemporal constraint:

$$\Sigma_{t=0}^{T}\Sigma_{m=1}^{M}\delta_t\rho_{tm}p^{tm}\cdot x^{tm}=W \tag{13}$$

where p^{tm} and q^{tm} are the (spot) price and quantity vectors for season m of t and x^{tm} is a year t, season m, year t vector of decision variables. Similarly, definition (4) for "wealth" W is now:

$$W\equiv\Sigma_{t=0}^{T}\Sigma_{m=1}^{M}\delta_t\rho_{tm}p^{tm}\cdot q^{tm}. \tag{14}$$

Making assumption (5) again, we can now derive the following counterparts to (6):

$$\begin{aligned} &max_{x^{t1},\ldots,x^{tM}}\{f(x^{t1},\ldots,x^{tM}):\Sigma_{m=1}^{M}\rho_{tm}p^{tm}\cdot x^{tm}=\Sigma_{m=1}^{M}\rho_{tm}p^{tm}\cdot q^{tm}\} \\ &=f(q^{t1},\ldots,q^{tM})\equiv f(q^t);\quad t=0,1,\ldots,T \end{aligned} \tag{15}$$

where the annual year t observed quantity vector q^t is equal to $[q^{t1},\ldots,q^{tM}]$ and q^{tm} is the season m, year t observed quantity vector.

Note that the seasonal discount factors ρ_{tm} appear in the constraints of the annual utility maximization problems (15). Define the vector of year t, season m discounted (to the beginning of year t) prices p^{tm*} as

$$p^{tm*}\equiv\rho_{tm}p^{tm};\quad t=0,1,\ldots,T;\quad m=1,\ldots,M. \tag{16}$$

The constraints in (15) can now be written as $p^{t*} \cdot x^t = p^{t*} \cdot q^t$ where the year t discounted price vector is defined as $p^{t*} \equiv [p^{t1*}, p^{t2*}, \ldots, p^{tM*}]$. Now we can repeat the analysis in the previous section associated with equations (7) - (12): we need only replace the year t spot price vectors p^t by the year t discounted vectors p^{t*}. In particular, assuming that the bilateral index number formula Q is exact for the homogeneous aggregator function f and its dual unit cost function c, we have the following counterparts to (7) and (11):

$$f(q^t)/f(q^s) = Q(p^{s*}, p^{t*}, q^s, q^t); \quad 0 \leq s, t \leq T; \tag{17}$$

$$c(p^{t*})/c(p^{s*}) = P(p^{s*}, p^{t*}, q^s, q^t); \quad 0 \leq s, t \leq T \tag{18}$$

where P is the bilateral price index associated with the quantity index Q defined using the counterpart to (9) which replaces p^s and p^t by p^{s*} and p^{t*}.

Thus our approach to constructing annual index numbers when there are seasonal commodities and high inflation is to use the Mudgett-Stone annual indexes with the year t season m spot prices p^{tm} replaced by the within year inflation adjusted prices p^{tm*} defined by (16).

To see why we must use inflation adjusted prices in our annual index number formulae, consider the situation when there is a hyperinflation and we are using the Fisher quantity index defined by (8). If the hyperinflation takes place only in season m of year t, then the Paasche part $p^t \cdot q^t/p^t \cdot q^s$ of the Fisher index will be approximately equal to $p^{tm} \cdot q^{tm}/p^{tm} \cdot q^{sm}$; i.e., only consumption in season m of year t, q^{tm}, and consumption in season m of year s, q^{sm}, will enter into the comparison between *years* s and t if spot prices p^{tm} are used in place of the discounted prices p^{tm*}. This is obviously undesirable.

Note that $\rho_{tm+1}/\rho_{tm} \equiv 1 + r_{tm}$ for $m = 1, 2, \ldots, M - 1$ where r_{tm} is the average interest rate faced when borrowing or lending money from the middle of season m to $m + 1$ in t. If prices are expected to increase in $m + 1$ compared to m, then the nominal interest rate r_{tm} can be expected to increase too.[18] Thus if the discounted prices $\rho_{tm}p_n^{tm}$ are used in place of the nominal prices p_n^{tm} in an annual index number formula, the effects of high inflation in any season will be nullified by the discount rates ρ_{tm}.

The use of the seasonally discounted prices p^{t*} in (17) and (18) in place of the nominal prices p^t poses difficulties for economic statisticians. Not only must the Statistical Agency collect seasonal data on nominal prices and quantities, but data on season to season interest rates r_{tm} must also be collected in order to calculate the seasonal discount factors ρ_{tm}. In principle, the interest rate r_{tm} should be a weighted average of all interest rates that consumers face (both borrowing and lending rates) where the weights are proportional to the amounts of funds loaned out or borrowed by consumers during season m of year t. This is not a trivial task. Moreover, many statisticians will object to using discounted prices in constructing annual price and quantity indexes on the grounds that the Fisher (1930)–Hicks (1946) intertemporal consumer theory that (17) and (18) are based on is too "unrealistic". Thus we consider some alternatives to the use of interest rates as discount factors in forming the seasonally deflated prices p^{tm*} defined by (16).

A simple alternative is to use the price of a widely traded commodity as a discount factor. Thus if p_G^{tm} is the price of gold in season m of year t, then the "gold standard" discount factors are:

$$\rho_{tm}^G \equiv p_G^{t1}/p_G^{tm}; \quad t = 0, 1, \ldots, T; \quad m = 1, \ldots, M. \tag{19}$$

The gold deflated prices $p_n^{tm*} \equiv \rho_{tm}^G p_n^{tm}$ could be used as the normalized prices in (17) and (18).[19]

Another alternative is to convert nominal prices into prices expressed in terms of a stable currency.[20] In this case, the "foreign currency" discount factors ρ_{tm}^E are defined by

$$\rho_{tm}^E \equiv e_{tm}/e_{t1}; \quad t = 0, 1, \ldots, T; \quad m = 1, \ldots, M, \tag{20}$$

where e_{tm} is the average number of units of foreign currency required to buy 1 unit of domestic currency in season m of year t.

Instead of using the price of gold p_G^{tm} as the deflator in (19), we could use any of the price for any commodity is traded during each season. Or instead of deflating by a single commodity price, the price or cost of a basket of nonseasonal and type (ii) (a) seasonal commodities might be used as the deflator. The season m year t price vector p^{tm} could be divided up into the vectors $[\tilde{p}^{tm}, \hat{p}^{tm}]$ where $\tilde{p}^{tm} \equiv [\tilde{p}_1^{tm}, \tilde{p}_2^{tm}, \ldots, \tilde{p}_K^{tm}]$ where each of the K commodities represented in $\tilde{p}^{tm}$ is either a nonseasonal commodity or a type (ii) (a) seasonal commodity.[21] Let $b \equiv [b_1, b_2, \ldots, b_K]$ be a vector of "appropriate" commodity quantity weights. Then the year t season m price of this basket of goods is $\tilde{p}^{tm} \cdot b$ and the "commodity standard" discount factors are defined by[22]

$$\rho^B_{tm} \equiv \tilde{p}^{tm} \cdot b / \tilde{p}^{tm} \cdot b; \quad t = 0, 1, \ldots, T; \quad m = 1, \ldots, M. \tag{21}$$

As a further refinement to (21), we could replace the fixed basket index $\tilde{p}^{tm} \cdot b$ by a general price index, $\tilde{P}(\tilde{p}^{t1}, \tilde{p}^{tm}, \tilde{q}^{t1}, \tilde{q}^{tm})$, which compares the prices of commodities (excluding type (i) and type (ii) (b) seasonal commodities) in season m of year t, $\tilde{p}^{tm}$, to their prices in the base period, $\tilde{p}^{t1}$. Now the discount factor is

$$\rho^P_{tm} \equiv 1/\tilde{P}(\tilde{p}^{t1}, \tilde{p}^{tm}, \tilde{q}^{t1}, \tilde{q}^{tm}); \quad t = 0, 1, \ldots, T; \quad m = 1, \ldots, M. \tag{22}$$

We will pursue this final refinement in section 7 below.

Each of the choices for the seasonal discount factors ρ_{tm} represented by (19) - (22) has advantages and disadvantages. All of these choices seem somewhat arbitrary. However, each of these will lead to sensible index number comparisons in the presence of hyperinflation. If we make use of Fisher's (1896; 69) observation that nominal rates of interest are approximately equal to real rates plus the rate of inflation, it can be seen that the inflation rate choices that are imbedded in the discount factor choices (19) - (22) will be approximately equal to our interest rate choice for ρ_{tm} that we advocated originally, provided that the season to season real rates of return are small.

The important conclusion that we should draw from the analysis presented in this section is that when constructing annual quantity indexes in high inflation situations, seasonal prices *must* be deflated for general inflation that occurred from season to season throughout the year. If this deflation is not done, the quantities corresponding to high inflation seasons will receive undue weight in the annual quantity index.

We conclude this section by discussing the interpretation of the annual price index $P(p^{s*}, p^{t*}, q^s, q^t)$ in (18). We assume that the price and quantity indexes P and Q that appear in (17) and (18) satisfy the weak factor reversal test (9) with normalized prices p^{t*} used in place of nominal prices p^t. Thus P and Q satisfy

$$\Sigma_{m=1}^{M} p^{tm*} \cdot q^{tm} / \Sigma_{m=1}^{M} p^{sm*} \cdot q^{sm} = P(p^{s*}, p^{t*}, q^s, q^t) Q(p^{s*}, p^{t*}, q^s, q^t); \tag{23}$$

i.e., using our original seasonal interest rate discount factors ρ_{tm}, (23) says that the discounted (to the beginning of year t) sum of seasonal values $\Sigma_{m=1}^{M} p^{tm*} \cdot q^{tm}$ divided by the discounted (to the beginning of year s) sum of seasonal values $\Sigma_{m=1}^{M} p^{sm*} \cdot q^{sm}$ is decomposed into $P(p^{s*}, p^{t*}, q^s, q^t)$ times $Q(p^{s*}, p^{t*}, q^s, q^t)$. The price index $P(p^{s*}, p^{t*}, q^s, q^t)$ captures the change in discounted year t prices relative to discounted year s prices. The interpretation of $P(p^{s*}, p^{t*}, q^s, q^t)$ when the specific commodity discount factors defined by (19) - (22) are used is less clear. If we use the discount factors defined by (19), then the normalized prices in season 1 of each year t, p_n^{t1*}, are equal to the corresponding nominal prices p_n^{t1}, but the normalized prices for later seasons $m > 1$, p_n^{tm*}, are equal to the corresponding nominal prices, p_n^{tm}, divided by the year t, season m to 1, gold price relative, p_G^{tm}/p_G^{t1}. $P(p^{s*}, p^{t*}, q^s, q^t)$ is a measure of price level change going from year s to t with the seasonal prices within each year are "stabilized" in terms of season 1 prices using the price of gold as the deflator of post season 1 prices. This index does not have a clear interpretation as a measure of the average level of nominal prices in year t versus year s.

In the following section, we discuss the possible use of annual unit values as prices in the construction of annual price and quantity indexes.

4. Annual Unit Value Indexes Under Conditions of High Inflation

The reader may well feel that the annual index number model that we developed in the previous section was too complex. One simpler alternative is the following: instead of distinguishing commodities by season, add up consumption of each physically distinct commodity over the seasons and use these annual total consumptions as the quantities to be inserted into an index number formula. The price corresponding to each such annual quantity would be the total annual value of expenditures on that physical commodity divided by the annual quantity–an annual unit value.

This is a reasonable proposal, particularly when we consider that at some stage of disaggregation, unit values must be used in order to aggregate up individual transactions, if we want to apply bilateral index number theory.[23] However, an important characteristic of a unit value is the time period over which it is calculated. As Fisher (1922; 318), Hicks (1946; 122) and Diewert (1995; 22) noted, the time period should be short enough so that individual variations of price within the period can be regarded as unimportant. In periods of rapid inflation or hyperinflation, nominal prices vary substantially between seasons[24]. Seasonal values that correspond to high inflation seasons will be weighted too heavily in the annual unit value.

The above argument does not rule out the use of annual unit values provided that nominal prices p_n^{tm} are replaced by the within the year inflation adjusted normalized prices p_n^{tm*} defined by (16), *and* provided that these prices are approximately constant across seasons m for each commodity n. This proviso will not be satisfied if there are seasonal commodities.

The problem with the use of (normalized) annual unit values when there are seasonal commodities can be illustrated as follows.[25] Imagine two years, where in the second year, after transportation and storage improvements, a constant quantity of a seasonal fruit, say bananas, is consumed at a constant price. In the first year, the same total annual quantity is consumed mostly in one season at a price slightly lower than the second year constant price. In the other seasons of the first year, one banana is consumed at a very high price. The prices are such that the value of banana consumption is constant over the two years. The unit value for bananas would also be constant over the two years as would the corresponding total annual quantity index. However, most economists would feel that the utility of banana consumption is higher in the second year compared to the first year and an index number comparison ought to show this. Given low seasonal real interest rates, under the above conditions the use of a Mudgett-Stone Fisher ideal quantity index would lead to a banana quantity index greater than 1. Thus there will generally be real biases in using annual (normalized) unit value indexes if there are substantial seasonal fluctuations in quantities and (normalized) prices.

In order to compare more formally the use of annual unit value indexes using normalized prices with the Mudgett-Stone annual indexes in the previous section, we will make the simplifying assumption that there are no type (i) and no type (ii) (b) seasonal commodities. Thus the dimensionality of the commodity space is constant over each season so that $N_m = N$ for $m = 1, \ldots, M$ and we can aggregate commodities over seasons.

Define the year t quantity for commodity n as the sum over the season m quantities:

$$Q_n^t \equiv \Sigma_{m=1}^M q_n^{tm}; \quad n = 1, \ldots, N; \quad t = 0, 1, \ldots, T. \tag{24}$$

Using the inflation adjusted normalized prices p_n^{tm*}, an annual normalized value for commodity n in year t is defined as

$$V_n^{t*} \equiv \Sigma_{m=1}^M p_n^{tm*} q_n^{tm}; \quad n = 1, \ldots, N; \quad t = 0, 1, \ldots, T. \tag{25}$$

The normalized unit value for good n is defined as

$$P_n^{t*} \equiv V_n^{t*}/Q_n^t; \quad n = 1, \ldots, N; \quad t = 0, 1, \ldots, T. \tag{26}$$

Define the year t vector of normalized unit values as $P^{t*} \equiv [P_1^{t*}, \ldots, P_N^{t*}]$ and the year t vector of total quantities consumed as $Q^t \equiv [Q_1^t, \ldots, Q_N^t]$ for $t = 0, 1, \ldots, T$.

The annual price and quantity vectors P^{t*} and Q^t can be used in calculating annual quantity indexes. We want to justify the use of such an index. We assume intertemporal utility function satisfies the assumptions (5). One of these which appears to be necessary for total annual year t quantities $Q^t = \Sigma_{m=1}^{M} q^m$ to solve (15) is

$$f(x^1, x^2, \ldots, x^M) = g(\Sigma_{m=1}^{M} x^m) \tag{27}$$

where g is an increasing, concave and linearly homogeneous function of N variables.[26] However, to ensure that the quantity vectors $[q^{t1}, \ldots, q^{tM}]$ are solutions to (15) when f is defined by (27), we also require equality of the normalized price vectors; i.e., we require[27]

$$p^{t1*} = p^{t2*} = \ldots = p^{tM*}; \quad t = 0, 1, \ldots, T. \tag{28}$$

To see why this is so, rewrite (15) when f is defined by (27) as follows:

$$\begin{aligned} &max_{x^1,\ldots,x^M}\{g(\Sigma_{m=1}^{M} x^m) : \Sigma_{m=1}^{M} p^{tm*} \cdot x^m = \Sigma_{m=1}^{M} p^{tm*} \cdot q^{tm}\} \\ &= g(\Sigma_{m=1}^{M} q^{tm}), \quad t = 0, 1, \ldots, T. \end{aligned} \tag{29}$$

If (28) were not true for some t, then in (29), we would find that *all* of the seasonal purchases in year t for any commodity where unequal prices prevailed would have to be concentrated in the seasons with the lowest prices, which would contradict the observed data.

Assuming that (27) and (28) are satisfied, we can apply exact index number theory and derive the following annual index number equalities:

$$g(Q^t)/g(Q^s) = Q^*(P^{s*}, P^{t*}, Q^s, Q^t); \quad 0 \le s,\, t \le T \tag{30}$$

for any index number formula Q^* that is exact for the annual aggregator function g. Thus we have provided an economic justification for the use of annual normalized unit values P^{t*} and total annual quantities Q^t in an index number formula.

Suppose that Q^* in (30) and Q in (17) are both Fisher ideal quantity indexes. Under what conditions will the annual unit value approach (which leads to (30) with $Q^* = Q_F^*$) give us the same *numerical* answer as the less restrictive Mudgett-Stone approach (which leads to (17) with $Q = Q_F$)?

Using definitions (24) - (26), it is easy to see that

$$p^{t*} \cdot q^t = \Sigma_{m=1}^{M} p^{tm*} \cdot q^{tm} = P^{t*} \cdot Q^t; \quad t = 0, 1, \ldots, T. \tag{31}$$

Hence a Fisher ideal index used in (17) will equal a Fisher ideal index used in (30); i.e.,

$$Q_F^*(P^{s*}, P^{t*}, Q^s, Q^t) = Q_F(p^{s*}, p^{t*}, q^s, q^t); \quad 0 \le s,\, t \le T, \tag{32}$$

if and only if[28]

$$P^{s*} \cdot Q^t = p^{s*} \cdot q^t \quad \text{for} \quad 0 \le s,\, t \le T. \tag{33}$$

A simple set of conditions that will ensure the equalities in (33) are the following Leontief (1936) type aggregation conditions:

$$q^{tm} = \alpha_t \beta_m \bar{q}; \quad t = 0, 1, \ldots, T; \quad m = 1, \ldots M \tag{34}$$

where $\alpha_t > 0$ is a year t growth factor, $\beta_m > 0$ is a shift factor for season m and $\bar{q} \equiv [\bar{q}_1, \ldots, \bar{q}_N]$ is a fixed quantity vector. If the β_m form an increasing sequence, they may be interpreted as "monthly" growth factors. If the β_m fluctuate with mean 1, they can be interpreted as pure seasonal fluctuation factors with *all* commodities subject to the *same* pattern of fluctuations.

We now verify that assumptions (34) imply the equalities (33). Using the definition of an inner product, we have for $0 \leq s, t \leq T$:

$$
\begin{aligned}
P^{s*} \cdot Q^t &= \Sigma_{n=1}^N P_n^{s*} Q_n^t \\
&= \Sigma_{n=1}^N [\Sigma_{m=1}^M p_n^{sm*} q_n^{sm} / \Sigma_{j=1}^M q_n^{sj}][\Sigma_{i=1}^M q_n^{ti}] \quad \text{using definitions (24) - (26)} \\
&= \Sigma_{n=1}^N [\Sigma_{m=1}^M p_n^{sm*} \alpha_s \beta_m \bar{q}_n / \Sigma_{j=1}^M \alpha_s \beta_j \bar{q}_n][\Sigma_{i=1}^M \alpha_t \beta_i \bar{q}_n] \quad \text{using (34)} \\
&= \Sigma_{n=1}^N \Sigma_{m=1}^M p_n^{sm*} \alpha_t \beta_m \bar{q}_n \\
&= \Sigma_{n=1}^N \Sigma_{m=1}^M p_n^{sm*} q_n^{tm} \quad \text{using (34)} \\
&= p^{s*} \cdot q^t
\end{aligned}
$$

where the last equality follows from the definitions of the annual vectors p^{s*} and q^t.

Thus assumptions (34) do indeed imply the equality of the Fisher indexes in (32) but they are not consistent with the simultaneous existence of both seasonal and nonseasonal commodities or with the existence of nonconstant "monthly" growth rates.

Another set of conditions that will ensure that the equalities in (33) hold are the following Hicks (1946; 312) aggregation conditions:

$$p^{tm*} = \gamma_t \bar{p}; \quad t = 0, 1, \ldots, T; \quad m = 1, \ldots, M \tag{35}$$

where $\gamma_t > 0$ is a year t price level factor and $\bar{p} \equiv [\bar{p}_1, \ldots, \bar{p}_N]$ is a constant price vector.

We now verify that assumptions (35) imply the equalities (33). Using definitions (24) - (26) again, we have for $0 \leq s,\, t \leq T$:

$$\begin{aligned}
P^{s*} \cdot Q^t &= \Sigma_{n=1}^N [\Sigma_{m=1}^M p_n^{sm*} q_n^{sm} / \Sigma_{j=1}^M q_n^{sj}][\Sigma_{i=1}^M q_n^{ti}] \\
&= \Sigma_{n=1}^N [\Sigma_{m=1}^M \gamma_s \bar{p}_n q_n^{sm} / \Sigma_{j=1}^M q_n^{sj}][\Sigma_{i=1}^M q_n^{ti}] \quad \text{using (35)} \\
&= \Sigma_{n=1}^N [\gamma_s \bar{p}_n][\Sigma_{i=1}^M q_n^{ti}] \\
&= \Sigma_{n=1}^N \Sigma_{m=1}^M \gamma_s \bar{p}_n q_n^{tm} \\
&= \Sigma_{n=1}^N \Sigma_{m=1}^M p_n^{sm*} q_n^{tm} \quad \text{using (35)} \\
&= p^{s*} \cdot q^t.
\end{aligned}$$

Thus conditions (35) imply the equalities in (33) and (32). Note that conditions (35) are just a different way of writing our earlier conditions (28). These conditions are very restrictive: they require absolute equality of all discounted seasonal price vectors p^{tm*} within each year t. In particular, these conditions rule out seasonal fluctuations in prices.

The above analysis indicates that the existence of seasonal commodities will generally cause the annual unit value index numbers to differ (perhaps substantially) from the Mudgett-Stone annual indexes studied in the previous two sections. Since the assumptions on the underlying annual aggregator function needed to derive exact indexes are much less restrictive for the Mudgett-Stone indexes, we recommend the use of these indexes over the use of annual unit value indexes.

We turn now to the task of justifying the use of season specific year over year indexes.

5. Year Over Year Seasonal Indexes

The separability assumptions on the annual aggregators function f which appears in (5) that are required to justify year over year seasonal indexes can be phrased as follows: there exists an increasing continuous function h of M variables and there exist functions f^m of N_m variables, $m = 1, \ldots, M$, such that

$$f(x^1, \ldots, x^M) = h[f^1(x^1), \ldots, f^M(x^M)] \tag{36}$$

where the seasonal aggregator $f^m(x^m)$ are increasing, linearly homogeneous and concave.

Assumption (36) says that the annual aggregator f which appeared in sections 2 and 3 above now has a more restrictive form which aggregates the seasonal vectors x^m in two stages. In the first, the commodities in season $m, x^m \equiv [x_1^m, x_2^m, \ldots, x_{N_m}^m]$, are aggregated by the season specific utility function $f^m(x^m) \equiv u_m$ and then the seasonal utilities u_m are aggregated in the second stage by h to form annual utility, $u \equiv h(u_1, u_2, \ldots, u_M)$.

Making assumption (5) again and assuming that the consumer's intertemporal budget constraint is defined by (13) and (14), we can again derive (15). Substituting (36) into (15) and using the assumption that h is increasing in its arguments gives:

$$max_{x^m}\{f^m(x^m) : p^{tm} \cdot x^m = p^{tm} \cdot q^{tm}\} = f^m(q^{tm});\ t = 0, 1, \ldots, T;\ m = 1, \ldots, M. \quad (37)$$

Let the unit cost dual c^m to the seasonal aggregator function f^m be defined by:

$$c^m(p^m) \equiv min_{x^m}\{p^m \cdot x^m : f^m(x^m) = 1\}; \quad m = 1, \ldots, M. \quad (38)$$

Let P^m and Q^m be price and quantity indexes that are exact for the season m aggregator function f^m. Then under our optimizing assumptions, we have the following equalities, applying the usual theory of exact index numbers, for $0 \leq s,\ t \leq T$ and $m = 1, \ldots, M$:

$$f^m(q^{tm})/f^m(q^{sm}) = Q^m(p^{sm}, p^{tm}, q^{sm}, q^{tm}); \quad (39)$$

$$c^m(p^{tm})/c^m(p^{sm}) = P^m(p^{sm}, p^{tm}, q^{sm}, q^{tm}). \quad (40)$$

Equation (39) says that the ratio of seasonal utility in season m of to seasonal utility in the same season m of s is equal to the quantity index $Q^m(p^{sm}, p^{tm}, q^{sm}, q^{tm})$ which is a function of the nominal price vectors for season m of s t, p^{sm} and p^{tm}, and the observed quantity vectors for season m of s and t, q^{sm} and q^{tm}. If the seasonal aggregator functions are chosen to be the flexible homogeneous quadratic functions $f^m(x^m) \equiv [x^m \cdot A^m x^m]^{1/2}$, where A^m is a square symmetric matrix of constants for $m = 1, \ldots, M$, then the corresponding exact Q^m and P^m will be the superlative Fisher ideal indexes Q_F^m and P_F^m for $m = 1, \ldots, M$.

Equation (40) tells us that the theoretical Konüs (1924) price index for season m between years s and t, $c^m(p^{tm})/c^m(p^{sm})$, is exactly equal to the price index $P^m(p^{sm}, p^{tm}, q^{sm}, q^{tm})$ which in turn will equal the Fisher ideal price index $P_F^m(p^{sm}, p^{tm}, q^{sm}, q^{tm})$ if f^m is the homogeneous quadratic aggregator function defined above.

Note that the nominal price vectors for season m in s and t, p^{sm} and p^{tm}, appear in (40). Thus the index number on the right hand side of (40) is a valid indicator of the amount of nominal price change that has occurred going from season m of s to the same season m in t. Presumably, we have chosen the seasons to be short enough so that prices can be assumed to be approximately constant within each season and hence have avoided the weighting problems encountered in the previous two sections in constructing annual indexes.

Summarizing the results of this section, we have shown how the separability assumption (36) justifies the use of the year over year seasonal price and quantity indexes that appeared in (39) and (40). These year over year seasonal indexes have been proposed by Flux (1921; 184), Zarnowitz (1961; 266) and many others[29] but explicit economic justifications for these indexes seem to be lacking.

In the following section, we ask whether the year over year seasonal indexes, (39) and (40), can be used as building blocks in the construction of annual indexes.

6. Consistency of Year Over Year Seasonal Indexes With An Annual Index

Recall the results of section 3 and specialize them so that the year s which appears in (17) and (18) is the base year, year 0. Let f be the linearly homogeneous, concave and increasing annual aggregator function which appears in (15) and (17) and let Q and P be exact for f. Then using (17) with $s = 0$, we have for $t = 0, 1, \ldots, T$:

$$f(q^t)/f(q^0) = f(q^{t1}, \ldots, q^{tM})/f(q^{01}, \ldots, q^{0M}) = Q(p^{0*}, p^{t*}, q^0, q^t). \tag{41}$$

Equations (41), along with a base period normalization for $f(q^0)$ such as $f(q^0) \equiv p^{0*} \cdot q^0$, can be used to compute the annual quantity aggregates $f(q^t)$ by utilizing the index number formula Q that is exact for f.

Now consider the model in the previous section where the annual aggregator function f had the more restrictive separable functional form defined by (36). How can the annual aggregates $f(q^t) = h[f^1(q^1), \ldots, f^M(q^M)]$ be computed exactly in this case?[30]

As in the previous section, assume that the seasonal aggregators $f^m(x^m)$ are linearly homogeneous, increasing and concave in their arguments and assume now that h has the same mathematical properties. Assume also that the f^m have exact index number formulae P^m and Q^m. We can again derive the equalities (39) and (40) and we can also derive the following counterparts to (39) and (40) (with $s = 0$)[31] where normalized prices p^{tm*} replace the nominal price vectors p^{tm}, for $t = 0, 1, \ldots, T$ and $m = 1, \ldots, M$:

$$f^m(q^{tm})/f^m(q^{0m}) = Q^m(p^{0m*}, p^{tm*}, q^{0m}, q^{tm}); \tag{42}$$

$$c^m(p^{tm*})/c^m(p^{0m*}) = P^m(p^{0m*}, p^{tm*}, q^{0m}, q^{tm}). \tag{43}$$

Choose units of measurement to measure base period seasonal utilities $f^m(q^{0m})$ as follows:

$$f^m(q^{0m}) \equiv p^{0m*} \cdot q^{0m} \equiv Q_m^0; \quad m = 1, \ldots, M; \tag{44}$$

i.e., set utility in season m of year 0, $f^m(q^{0m})$ or Q_m^0, equal to base period expenditures in $m, p^{0m} \cdot q^{0m}$, times the inflation factor ρ_{0m} which converts the dollars of season m in year 0 to dollars at the beginning of year 0; (remember that $p^{0m*} = \rho_{0m} p^{0m}$). The normalizations (44) imply that base year seasonal unit costs, $c^m(p^{0m*})$, are all equal to unity; i.e.,

$$c^m(p^{0m*}) = 1 \equiv P_m^{0*}; \quad m = 1, \ldots, M. \tag{45}$$

We have used equations (44) and (45) to define Q_m^0 and P_m^{0*} for $m = 1, \ldots, M$. Now substitute (44) and (45) into (42) and (43) to obtain the following computable formulae for the year t seasonal price and quantity aggregates, $c^m(p^{tm*})$ and $f^m(q^t)$ for $t = 1, \ldots, T$ and $m = 1, \ldots, M$:

$$f^m(q^{tm}) = Q^m(p^{0m*}, p^{tm*}, q^{0m}, q^{tm}) p^{0m*} \cdot q^{0m} \equiv Q_m^t; \tag{46}$$

$$c^m(p^{tm*}) = P^m(p^{0m*}, p^{tm*}, q^{0m}, q^{tm}) \qquad \equiv P_m^{t*}. \tag{47}$$

Note that we have used equations (46) and (47) to define year t and season m seasonal price and quantity aggregates, P_m^t and Q_m^t.

Now consider the year t utility maximization problems (15) when f has the separable form (36) for $t = 0, 1, \ldots, T$:

$$max_{x^1,\ldots,x^M}\{h[f^1(x^1),\ldots,f^M(x^M)] : \Sigma_{m=1}^M p^{tm*}\cdot x^m = \Sigma_{m=1}^M p^{tm*}\cdot q^{tm}\}$$

$$= max_{x^1,\ldots,x^M}\{h[f^1(x^1),\ldots,f^M(x^M)] : \Sigma_{m=1}^M c^m(p^{tm*})f^m(x^m) = \Sigma_{m=1}^M c^m(p^{tm*})f^m(q^{tm})\}$$

since maximization of utility implies cost minimization[32]

$$= max_{Q_1,\ldots,Q_M}\{h[Q_1,\ldots,Q_m] : \Sigma_{m=1}^M P_m^{t*}Q_m = \Sigma_{m=1}^M P_m^{t*}Q_m^t\}$$

letting $Q_m \equiv f^m(x^m), Q_m^t \equiv f^m(q^{tm})$ and $P_m^{t*} \equiv c^m(p^{tm*})$

$$= h[Q_1^t,\ldots,Q_M^t]. \tag{48}$$

The equalities in (48) follow from the assumption that the observed quantity data for year $t, q^t \equiv [q^{t1},\ldots,q^{tM}]$, solve the year t utility maximization problem (15) when f has the separable structure (36) and the homogeneous seasonal aggregator functions f^m have the exact index number formulae $Q^m(p^{0m*},p^{tm*},q^{0m},q^{tm})$ and $P^m(p^{0m*},p^{tm*},q^{0m},q^{tm})$ that enabled us to construct the seasonal price and quantity aggregates P_m^{t*} and Q_m^t via (44) - (47).

Let the annual aggregate quantity index Q^a be exact for the linearly homogeneous aggregator function h. Then the equalities in (48) imply the following exact index number relationships for $t = 1, 2, \ldots, T$:

$$\begin{aligned} &h[f^1(q^{t1}),\ldots,f^M(q^{tM})]/h[f^1(q^{01}),\ldots,f^M(q^{0M})] \\ &= Q^a(P_1^{0*},\ldots,P_M^{0*};P_1^{t*},\ldots,P_M^{t*};Q_1^0,\ldots,Q_M^0;Q_1^t,\ldots,Q_M^t). \end{aligned} \tag{49}$$

The index number formula on the right hand side of (49) is an example of a two stage aggregation formula. In the first, we use the year over year "monthly" indexes Q^m and P^m to form the "monthly" aggregate prices and quantities P_m^{t*} and Q_m^t using (44) - (47). In the second stage, the annual quantity index Q^a aggregates the "monthly" information using the right hand side of (49) to form an estimator for the ratio of real consumption in t versus 0.[33]

The two stage estimator of the annual consumption ratio defined by (49) can be compared with the single stage estimator defined by the right hand side of (41). In general, the two stage estimator (49) will not coincide with the one stage estimator (41). However, there are special cases of interest to Statistical Agencies where the two index number approaches will yield exactly the same answer: if all of the aggregator functions $f, f^1, \ldots, f^m$ and h are of the Leontief (1936) no substitution variety[34], then corresponding exact price and quantity indexes are (i) the Laspeyres price indexes $P_L, P_L^1, \ldots, P_L^M$ and P_L^a and the Paasche quantity indexes $Q_P, Q_P^1, \ldots, Q_P^M$ and Q_P^a, and (ii) the Paasche price indexes $P_P, P_P^1, \ldots, P_P^M$ and P_P^a and the Laspeyres quantity indexes $Q_L, Q_L^1, \ldots, Q_L^M$ and Q_L^a. Thus if Paasche or Laspeyres indexes are used throughout, then the year over year seasonal indexes can be used as building blocks in a two stage procedure to construct an annual index, and this procedure will give the same answer as the single stage procedure.

However, from the viewpoint of economic theory, the use of Paasche and Laspeyres indexes cannot be readily justified. The problem is that these indexes are exact only for Leontief aggregators which assume zero substitutability between all commodities.[35]

If we make the reasonable assumption that all of the homogeneous aggregators f, f^1, $\ldots, f^M$ and h can be closely approximated by homogeneous quadratic utility functions, then the corresponding exact index number formulae for Q, $Q^1, \ldots,$ $Q^M, P^1,$ $\ldots, P^M$ and Q^a are all (superlative) Fisher ideal indexes. In this case, the single stage annual aggregate quantity ratio defined by the right hand side of (41), $Q_F(p^{0*}, p^{t*}, q^0, q^t)$, will not be precisely equal to the corresponding two stage annual aggregate quantity ratio defined by the right hand side of (49) where Q^a is the Fisher ideal quantity index Q^a_F. However, Diewert (1978; 889), drawing results due to Vartia (1974) (1976), showed that, numerically, the right hand side of (41) will approximate the right hand side of (49) to the second order[36], provided that superlative index number formulae were used for all of the indexes. Limited empirical evidence on the closeness of single stage superlative indexes to their two stage counterparts can be found in Diewert (1978; 895) (1983c; 1036-1040).

If the single stage number differs considerably from the two stage number, which number should be used? If superlative indexes are being used in both procedures, then from the viewpoint of economic theory, the single stage number should be preferred, since the assumptions on the annual preference function f are the weakest using this procedure.

We turn now to the difficult problem of making index number comparisons between seasons within the same year when there are seasonal commodities.

7. Short Term Season To Season Indexes

Under conditions of even low inflation, it is important to have reliable short term inflation measures for indexation, wage negotiations, calculation of real rates of return, etc. Thus we need to be able to compare the price level of the current season with the immediately preceding ones. The annual price indexes defined earlier are not suitable for this nor are the year over year seasonal indexes defined by (40), since they are not comparable over seasons or "months" m because the commodity baskets change over the seasons due to the existence of type (i) seasonal commodities. To make this lack of comparability problem clearer, make the separability assumption (36) on the annual utility function f. Assume that the season m aggregator f^m has the unit cost dual c^m, and P^m is an exact bilateral price index for f^m. Setting $s = 0$, equations (40) become:

$$c^m(p^{tm})/c^m(p^{0m}) = P^m(p^{0m}, p^{tm}, q^{0m}, q^{tm}); \quad t = 1, \ldots, T; \quad m = 1, \ldots, M. \qquad (50)$$

We can interpret $c^m(p^{tm})$ as the price or unit cost of one unit of season m subutility in year t, but there is no way of comparing these subutilities across seasons.[37] Thus equations (50) are of no help in obtaining comparable (across seasons) price indexes.

The aboved lack of comparability problem was noted by Mudgett (1955; 97-98), Zarnowitz (1961; 246) and the economic statisticians at INSEE (1976; 67): the existence of type (i) seasonal goods makes it impossible to carry out normal bilateral index number comparisons between consecutive seasons.

A solution to this problem of a lack of comparability is to make a different separability assumption. Recall the notation from in section 3 where we partitioned the price vector p^{tm} into $[\tilde{p}^{tm}, \hat{p}^{tm}]$ where the commodities represented in $\tilde{p}^{tm}$ were either nonseasonal commodities or type (ii) (a) seasonal commodities. Partition the quantity vectors in a similar manner; i.e., $q^{tm} \equiv [\tilde{q}^{tm}, \hat{q}^{tm}]$ and $x^{tm} \equiv [\tilde{x}^{tm}, \hat{x}^{tm}]$ for $t = 0, 1, \ldots, T$ and $m = 1, \ldots, M$. We now assume that the intertemporal utility function U introduced in section 1 has the following structure: there exists an increasing, continuous function G and an increasing, linearly homogeneous and concave function ϕ such that

$$\begin{aligned} &U(x^{01}, \ldots, x^{0M}; \ldots; x^{T1}, \ldots, x^{TM}) \\ &\quad = G[\phi(\tilde{x}^{01}), \hat{x}^{01}, \ldots, \phi(\tilde{x}^{0M}), \hat{x}^{0M}; \ldots; \phi(\tilde{x}^{T1}), \hat{x}^{T1}, \ldots, \phi(\tilde{x}^{TM}), \hat{x}^{TM}]. \end{aligned} \tag{51}$$

The assumptions on the structure of intertemporal preferences represented by (51) are similar to the separability assumptions made by Pollak (1989; 77) to justify the usual annual indexes (recall sections 2 and 3 above). The only difference is that we now want to justify comparable "monthly" indexes and thus our "monthly" aggregator ϕ must not include type (i) and type (ii) (b) seasonal goods.[38]

Using our new notation for $p^{tm} \equiv [\tilde{p}^{tm}, \hat{p}^{tm}]$, $x^{tm} \equiv [\tilde{x}^{tm}, \hat{x}^{tm}]$ and $q^{tm} \equiv [\tilde{q}^{tm}, \hat{q}^{tm}]$, we can rewrite the consumer's intertemporal budget constraint given by (13) and (14) as:

$$\Sigma_{t=0}^{T} \Sigma_{m=1}^{M} \delta_t \rho_{tm} [\tilde{p}^{tm} \cdot \tilde{x}^{tm} + \hat{p}^{tm} \cdot \hat{x}^{tm}] = \Sigma_{t=0}^{T} \Sigma_{m=1}^{M} \delta_t \rho_{tm} [\tilde{p}^{tm} \cdot \tilde{q}^{tm} + \hat{p}^{tm} \cdot \hat{q}^{tm}]. \tag{52}$$

As usual, we assume that $[q^0, q^1, \ldots, q^T]$ solves the intertemporal utility maximization problem when U is defined by (51) and the budget constraint is defined by (52), where the year t observed quantity vector is $q^t \equiv [q^{t1}, \ldots, q^{tm}]$ and the year t season m quantity vector is $q^{tm} \equiv [\tilde{q}^{tm}, \hat{q}^{tm}]$. Using the assumptions that G and ϕ are increasing in their arguments, we can deduce that[39]

$$max_{\tilde{x}^{tm}}\{\phi(\tilde{x}^{tm}) : \tilde{p}^{tm} \cdot \tilde{x}^{tm} = \tilde{p}^{tm} \cdot \tilde{q}^{tm}\} = \phi(\tilde{q}^{tm}); \quad t = 0, 1, \ldots, T; \quad m = 1, \ldots, M. \tag{53}$$

Let $\gamma(\tilde{p}^{tm})$ be the unit cost function that is dual to the short run aggregator function ϕ. Assume that the bilateral price and quantity indexes $\tilde{P}$ and $\tilde{Q}$ are exact for the aggregator function ϕ. Then the equalities (53) imply the following equalities for $0 \leq s,\ t \leq T;\ m = 1, \ldots, M$ and $j = 1, 2, \ldots, M$:

$$\phi(\tilde{q}^{tm})/\phi(\tilde{q}^{sj}) = \tilde{Q}(\tilde{p}^{sj}, \tilde{p}^{tm}, \tilde{q}^{sj}, \tilde{q}^{tm}); \tag{54}$$

$$\gamma(\tilde{p}^{tm})/\gamma(\tilde{p}^{sj}) = \tilde{P}(\tilde{p}^{sj}, \tilde{p}^{tm}, \tilde{q}^{sj}, \tilde{q}^{tm}). \tag{55}$$

We normalize the theoretical "monthly" price level function $\gamma(\tilde{p}^{tm})$ so that the seasonal price level in season 1 of year 0 is unity; i.e., we place the following restriction on γ:

$$\gamma(\tilde{p}^{01}) = 1. \tag{56}$$

Equations (55) and the normalization (56) allow us to use the exact bilateral index number formula $\tilde{P}$ to provide estimates for the theoretical short term seasonal price levels $\gamma(\tilde{p}^{tm})$.

The fixed base sequence of short term inflation estimates is

$$1, \tilde{P}(\tilde{p}^{01}, \tilde{p}^{02}, \tilde{q}^{01}, \tilde{q}^{02}), \ldots, \tilde{P}(\tilde{p}^{01}, \tilde{p}^{0M}, \tilde{q}^{01}, \tilde{q}^{0M}); \ldots; \tilde{P}(\tilde{p}^{01}, \tilde{p}^{T1}, \tilde{q}^{01}, \tilde{q}^{T1}), \ldots, \tilde{P}(\tilde{p}^{01}, \tilde{p}^{TM}, \tilde{q}^{01}, \tilde{q}^{TM}). \tag{57}$$

Using the chain principle, the sequence of short run inflation estimates is

$$1, \tilde{P}(\tilde{p}^{01}, \tilde{p}^{02}, \tilde{q}^{01}, \tilde{q}^{02}), \tilde{P}(\tilde{p}^{01}, \tilde{p}^{02}, \tilde{q}^{01}, \tilde{q}^{02})\tilde{P}(\tilde{p}^{02}, \tilde{p}^{03}, \tilde{q}^{02}, \tilde{q}^{03}), \ldots\ . \tag{58}$$

The first two numbers in the chain sequence (58) coincide with the first two numbers in the fixed base sequence (57) but then the chain estimate for a given year t and month $m+1$ is equal to the chain estimate for the immediately preceding month m times the month to month bilateral link, $\tilde{P}(\tilde{p}^{m}, \tilde{p}^{m+1}, \tilde{q}^{m}, \tilde{q}^{m+1})$. There are other ways of utilizing the exact index number bilateral relationship defined by (55) to obtain estimates for the sequence of "month" to "month" theoretical price levels

$$\gamma(\tilde{p}^{01}), \gamma(\tilde{p}^{02}), \ldots, \gamma(\tilde{p}^{0M}); \ldots; \gamma(\tilde{p}^{T1}), \gamma(\tilde{p}^{T2}), \ldots, \gamma(\tilde{p}^{TM}); \tag{59}$$

but the fixed base and chain methods are the most practical ones.[40]

Should the fixed base sequence of price levels (57) or should the chain sequence (58) be used by Statistical Agencies to measure short term price change? It seems appropriate to review some of the arguments.[41]

One of the main arguments in favour of the chain system is that it is better adapted to solving the problems of disappearing and also new goods.[42] Recently, the problem of the proliferation of new goods has intensified. Hence, the Statistical Agency, in making “month” to “month” index number comparisons, will be forced to use what Keynes (1930; 95) called the highest common factor method: the bilateral index number formula $\tilde{P}$ is applied only to the subset of commodities that are transacted in both periods.[43] If the chain system is used, then the subset of commodities transacted will be larger than the subset obtained using the fixed base system. Hence the chain comparisons will be more reliable than the fixed base comparisons.

Another argument in favour of the chain system is that most “reasonable” index number formulae will more closely approximate each other if the chain system is used, since period to period changes are likely to be smaller over adjacent periods. Diewert (1978; 895) and Hill (1988; 143) (1993; 387-388) noted that chaining will tend to reduce the spread between the Paasche and Laspeyres indexes and hence the use of either of these should more closely approximate a superlative index and hence better approximate the underlying economic index.[44]

However, Hill (1988; 136-137), drawing on some analysis by Szulc (1983; 548) on the “bouncing” phenomenon, also made an argument that favours the use of the fixed base system: the chain system should *not* be used if prices and quantities have a tendency to oscillate in a regular fashion and hence should not be used to aggregate seasonal data. To see why the chain principle can give poor results, consider a situation where the price and quantity data of quarter 1 in year 0 coincides with the price and quantity data of quarter 1 in year $t > 1$. Then if $\tilde{P}$ satisfies the identity test, the fixed base price level for quarter 1 of t will be $\tilde{P}(\tilde{p}^{01}, \tilde{p}^{t1}, \tilde{q}^{01}, \tilde{q}^{t1}) = \tilde{P}(\tilde{p}^{01}, \tilde{p}^{01}, \tilde{q}^{01}, \tilde{q}^{t1}) = 1$, the correct answer, whereas the chain index will not in general give the correct answer.[45] However, in the present context, this criticism loses all or most of its force, since we are excluding most seasonal commodities in the index number formula $\tilde{P}$.[46] Furthermore, if we take $\tilde{P}$ to be the Fisher ideal price index $\tilde{P}_F$, then in most cases, we will find that the chain principle will give satisfactory results even if there are type (ii) (a) seasonal goods included in the list of goods that $\tilde{P}_F$ operates on.[47]

Although our focus in this section is on measuring short term price change using the bilateral price index $\tilde{P}$, we can also use the companion quantity index $\tilde{Q}$ to measure short term quantity change for nonseasonal quantities. Furthermore, the exact index number relations (54), along with a base period normalization such as

$$\phi(\tilde{q}^{01}) = \tilde{p}^{01} \cdot \tilde{q}^{01} \tag{60}$$

which sets season 1 utility in the base year 0 equal to expenditure on nonseasonal goods $\tilde{p}^{01} \cdot \tilde{q}^{01}$, can be used to form estimates for annual sums of seasonal utilities. If we define year t aggregate utility by $\Sigma_{m=1}^{M} \phi(\tilde{q}^{tm})$, then using the fixed base principle, this theoretical real quantity aggregate can be estimated in units of season 1 year 0 constant dollars by

$$[\Sigma_{m=1}^{M} \tilde{Q}(\tilde{p}^{01}, \tilde{p}^{tm}, \tilde{q}^{01}, \tilde{q}^{tm})]\tilde{p}^{01} \cdot \tilde{q}^{01}. \tag{61}$$

The reader can work out chain system or multilateral estimates for the year t utility aggregate. However, annual quantity estimates of the form (61) will be of limited interest due to the exclusion of type (i) and type (ii) (b) seasonal goods. To obtain comprehensive estimates, it will be necessary to use the Mudgett-Stone indexes described in section 2 (with low inflation) or section 3 (with high or moderate inflation).

Our discussion can be summarized as follows: (1) a "month" to "month" Fisher ideal chain index of nonseasonal commodities (and of type (ii) (a) seasonal commodities) is our preferred alternative; see (58) with $\tilde{P} \equiv \tilde{P}_F$; (2) if quantity information is not available in a timely manner, fixed base Laspeyres price indexes will have to be used; i.e., (57) will have to be used with $\tilde{P} \equiv \tilde{P}_L$. However, the base period should be changed as frequently as possible.

Some seasonal bilateral index number procedures that work over commodity spaces of varying dimensions have been proposed by Diewert (1980; 506-508) and Balk (1980a; 27) (1981). We shall now review these proposals and compare them to our preferred proposal (58), which depended on the separability assumptions (51).

Diewert (1980; 507) attempted to deal with the problem of disappearing and then reappearing seasonal goods by utilizing Hicks' (1940; 114)[47] treatment of new goods: in seasons when a good is unavailable, determine the reservation price that would just ration the consumer's demand for the good down to zero. These reservation prices, along with the associated zero quantities, could then be used as prices and quantities that could be inserted into a bilateral season to season index number formula. There are two problems with this proposal: (1) Statistical Agencies do not have the resources required to estimate these reservation prices[48] and (2) even if appropriate reservation prices could be estimated, the assumptions required to justify the economic approach would not generally be satisfied. On the second problem, recall our earlier classification of type (ii) seasonal goods (price and quantity data are available in each season) into types (ii) (a) and (ii) (b). Once we have estimated reservation prices for type (i) seasonal goods, we have essentially converted them into type (ii) seasonal goods. Hence we have the same problem that we had with type (ii) seasonal commodities–some type (i) seasonal commodities can have their prices and quantities rationalized by maximizing an underlying utility aggregator function over the seasons (call these type (i) (a) seasonal commodities) and some cannot, because custom shifts the aggregator function over the seasons (call these type (i) (b) seasonal commodities). Thus to rigorously justify Diewert's (1980; 507) earlier economic approach, we have to rule out type (i) (b) and type (ii) (b) seasonal commodities, or simply restrict the index number comparisons to type (i) (a) and type (ii) (a) seasonal commodities. But this latter case is essentially the case analyzed in this section, except that we now add type (i) (a) commodities to our list of K type (a) seasonal commodities (and we have to provide reservation prices for the type (i) (a) commodities).

Balk's (1980a; 27) (1981) proposal for dealing with type (i) seasonal commodities makes use of the Vartia II (1974; 70) (1976) price index[49] so it is necessary to define this. First define the logarithmic mean[50] of two positive numbers, x and y, by

$$L(x,y) \equiv \begin{cases} [x-y]/[lnx - lny] & \text{if} \quad x \neq y \\ x & \text{if} \quad x = y. \end{cases} \tag{62}$$

Balk (1981; 73) observed that (62) could be extended to the case where one of the numbers x or y is zero and the other is positive so $L(x,y) = 0$. To define the Vartia II price index, let p^t be two generic price and quantity vectors pertaining to periods $t = 0, 1$. Define the period t expenditure share on commodity n by

$$w_n^t \equiv p_n^t q_n^t / p^t \cdot q^t; \quad t = 0,1; \quad n = 1, \ldots, N. \tag{63}$$

Define the logarithmic mean average share for commodity n between periods 0 and 1 by

$$w_n^{01} \equiv \begin{cases} L(w_n^0, w_n^1) & \text{if at least one of } w_n^0, w_n^1 \text{ is positive} \\ 0 & \text{if both } w_n^0 \text{ and } w_n^1 \text{ are } 0 \end{cases} \tag{64}$$

Finally, define the Sato (1976; 224)–Vartia (1974; 70) price index P_{SV} by

$$lnP_{SV}(p^0, p^1, q^0, q^1) \equiv \Sigma_{n=1}^N w_n^{01} ln(p_n^1/p_n^0) / \Sigma_{n=1}^N w_n^{01}. \tag{65}$$

We have added Sato's name to the price index P_{SV} defined by (65) because he showed that P_{SV} is exact for a constant elasticity of substitution (CES) aggregator function.[51]

Balk (1995b) and Reinsdorf and Dorfman (1995) have studied the axiomatic properties of the Sato-Vartia price index and their conclusion is that it almost rivals the Fisher ideal index.[52] Furthermore, the fact that the Sato-Vartia index is exact for CES functional forms has proved useful in empirical applications; e.g., see Feenstra (1994). However, it should be pointed out that the Sato-Vartia price index P_{SV} defined by (65) is *not* superlative; i.e., it is not exact for an aggregator function that can provide a second order approximation to an arbitrary twice differentiable linearly homogeneous function when the number of commodities N exceeds 2.[53]

We now return to Balk's (1980a; 27) (1981) proposal for dealing with seasonal commodities. His method works as follows: when comparing type (i) commodities between two seasons when both are absent from the marketplace, drop the commodity from the index number computation; for all other cases, use the Sato-Vartia price index. This procedure will set the weight of the commodity equal to zero if it is not present in both periods. Obviously, another way of describing this method is: use what Keynes (1930; 94) called the highest common factor method and use the Sato-Vartia price index as the index number formula.

Balk's approach[54] to the treatment of type (i) seasonal commodities is satisfactory from the viewpoint of the test approach to index number theory but there are two problems versus the economic approach: (1) since the Sato-Vartia index is not superlative, it would be better to apply the highest common factor method but use a superlative price index in place of the Sato-Vartia index[55] and (2) Balk's procedure ignores the existence of type (i) (b) and (ii) (b) seasonal commodities. The prices and quantities corresponding to these type (b) seasonal commodities cannot be rationalized by utility maximizing behavior where the utility function remains constant over the two periods in question because custom shifts the demand over seasons for type (b) seasonal commodities. The above second criticism of Balk's proposed procedure is the same as our earlier second criticism of Diewert's (1980; 507) economic approach to seasonal indexes and the cure to this problem is the same: restrict the season to season index number comparisons to nonseasonal commodities and type (i) (a) and (ii) (a) seasonal commodities.

To conclude this section, we note that many consumer goods are durable; i.e., they provide services beyond the initial season of purchase. Hence from the viewpoint of the economic approach to the short term consumer price index, seasonal rental prices or user costs should be used as the prices for durable consumer goods and the quantity weights should reflect not only the purchases made during the season but also the available stocks of consumer durables.[56] Note that as inflation increases, the "season" will generally have to shrink (so that within season price variation can be neglected) and thus the number of affected consumer durables will increase (and more user costs will have to be constructed).

In the following section, we return to the annual calendar year indexes defined in sections 2 and 3 above, but we no longer restrict ourselves to calendar years.

8. Moving Year Annual Indexes

In sections 2 and 3 above, the index number comparisons always compared the M seasons in one calendar year with the M seasons in another calendar year. However, we could choose any month (or season) as our year ending month and the prices and quantities of this new noncalendar year could be compared over years. The separability assumptions required to justify these new noncalendar year comparisons will be analogous to our earlier separability assumptions (5) but slightly different: the annual aggregator function f will now be defined over the seasonal commodity vectors for a noncalendar year. These noncalendar year comparisons can be taken a step further: we could think about comparing the prices and quantities of a *noncalendar* year with the prices and quantities of a base *calendar* year. What are the restrictions on intertemporal preferences that would justify this type of comparison, which we will call a variable year end comparison or a moving year[57] comparison? We provide an answer below.

Recall the seasonal aggregators $f^1(x^1), \ldots, f^M(x^M)$ from sections 5 and 6. We assumed the existence of an aggregator h that allowed us to define the annual utility function $f(x^1, \ldots, x^M) \equiv h[f^1(x^1), \ldots, f^M(x^M)]$. Here we again assume the existence of the linearly homogeneous, increasing and concave seasonal aggregators $f^1, \ldots, f^M$ but make the following stronger assumptions on the intertemporal utility function U:

$$U(x^{01}, \ldots, x^{0M}; x^{11}, \ldots, x^{1M}; \ldots; x^{T1}, \ldots, x^{TM}) \equiv \psi^{-1}\{\Sigma_{t=0}^{T} \Sigma_{m=1}^{M} \beta_m \psi[f^m(x^{tm})]\} \quad (66)$$

where the $\beta_m > 0$ are parameters that allow the consumer to cardinally compare the transformed seasonal utilities $\psi[f^m(x^{tm})]$ and $\psi(z)$ is a monotonic function of one positive variable z defined by

$$\psi(z) \equiv f_\alpha(z) \equiv \begin{cases} z^\alpha & \text{if } \alpha \neq 0 \\ lnz & \text{if } \alpha = 0. \end{cases} \tag{67}$$

Substituting (67) into (66) reveals that the intertemporal utility function U is a CES (constant elasticity of substitution) aggregate of the seasonal utilities $f^m(x^{tm})$. Using the assumptions that the seasonal aggregator functions $f^m(x^{tm})$ are linearly homogeneous in the elements of x^{tm}, it can be verified U is linearly homogeneous.[58]

Assume that $q^{01}, \ldots, q^{0M}; \ldots; q^{T1}, \ldots, q^{TM}$ solve the intertemporal utility maximization problem (2) where U is defined by (66) and (67) and the intertemporal budget constraint is defined by (13) and (14). Then since ψ^{-1} is a monotonic function of one variable[59], it can be seen that for any year t, we must have for $t = 0, 1, \ldots, T$:

$$\begin{aligned} \Sigma_{m=1}^M \beta_m \psi[f^m(q^{tm})] = max_{x^1,\ldots,x^M}\{\Sigma_{m=1}^M \beta_m \psi[f^m(x^m)] : \Sigma_{m=1}^M \delta_t \rho_{tm} p^{tm} \cdot x^m \\ = \Sigma_{m=1}^M \delta_t \rho_{tm} p^{tm} \cdot q^{tm}\}. \end{aligned} \tag{68}$$

Recall that $\delta_t > 0$ is the discount factor that makes a dollar at the beginning of t equivalent to a dollar at the beginning of 0. Recall also that p^{tm} is the vector of prices for season m of t and ρ_{tm} is the discount factor that makes a dollar in the middle of season m of t equivalent to a dollar at the beginning of year t.

In section 3 above, we defined the vector of year t, season m discounted (to the beginning of t) prices, p^{tm*}, by (16). In the present section when we will be dealing with noncalendar years, it is convenient to redefine p^{tm*} as follows:

$$p^{tm*} \equiv \delta_t \rho_{tm} p^{tm}; \quad t = 0, 1, \ldots, T; \quad m = 1, \ldots, M; \tag{69}$$

i.e., p^{tm*} is now the nominal price vector p^{tm} discounted to the beginning of 0.

Now return to the equalities (68). The annual utility $\Sigma_{m=1}^{M} \beta_m \psi[f^m(q^{tm})]$ can be rescaled or transformed by the monotonic function ψ^{-1} to make the resulting annual utility function linearly homogeneous. We obtain the equalities for $t = 0, 1, \ldots, T$:

$$\begin{aligned} &\psi^{-1}\{\Sigma_{m=1}^{M} \beta_m \psi[f^m(q^{tm})]\} \\ &\quad = max_{x^1,\ldots,x^M}\{\psi^{-1}(\Sigma_{m=1}^{M} \beta_m \psi[f^m(x^m)]) : \Sigma_{m=1}^{M} p^{tm*} \cdot x^m \\ &\qquad = \Sigma_{m=1}^{M} p^{tm*} \cdot q^{tm}\}. \end{aligned} \tag{70}$$

Recall equations (44) - (47). These remain valid now with the understanding that the normalized prices p^{tm*} are now defined by (69) instead of (16). The calendar year utility maximization problems in (70) are special cases of the year t utility maximization problems in (48), where the p^{tm*} are defined by (69). The general utility function h which appears in (48) is now the following specialized functional form:

$$\begin{aligned} h(Q_1, \ldots, Q_M) &\equiv \psi^{-1}[\Sigma_{m=1}^{M} \beta_m \psi(Q_m)] \\ &= [\Sigma_{m=1}^{M} \beta_m (Q_m)^{\alpha}]^{1/\alpha} \quad \text{if } \alpha \neq 0. \end{aligned} \tag{71}$$

Assuming that we have exact index number formulae for the seasonal aggregator functions $f^1, \ldots, f^M$, we can use (44) - (47) to calculate the seasonal (discounted to the beginning of year 0) prices P_m^{t*} and the seasonal aggregate quantities Q_m^t for $t = 0, 1, \ldots, T$ and $m = 1, \ldots, M$. Furthermore, (49) can be used to calculate the calendar year aggregates,

$h[f^1(q^{t1}),\ldots,f^M(q^{tM})]/h[f^1(q^{01}),\ldots,f^M(q^{0M})]$, provided that we can find the bilateral index number formula Q^a that is exact for the aggregator function h defined by (71). Note that the h defined by (71) has a CES (or mean of order α) functional form. Sato (1976; 225) showed that the Vartia II (1974; 66-70) (1976) quantity index Q_{SV} is exact for this functional form. Thus we have for $t = 1, 2, \ldots, T$:

$$
\begin{aligned}
&ln\{h[f^1(q^{t1}),\ldots,f^M(q^{tM})]/h[f^1(q^{01}),\ldots,f^M(q^{0M})]\} \\
&\quad = lnQ_{SV}(P_1^{0*},\ldots,P_M^{0*};P_1^{t*},\ldots,P_M^{t*};Q_1^0,\ldots,Q_M^0;Q_1^t,\ldots,Q_M^t) \qquad (72)\\
&\quad \equiv \Sigma_{m=1}^M w_m^{0t} ln(Q_m^t/Q_m^0)/\Sigma_{j=1}^M w_j^{0t}
\end{aligned}
$$

where $w_m^{0t} \equiv L(w_m^0, w_m^t), w_m^t \equiv P_m^{t*}Q_m^t/\Sigma_{j=1}^M P_j^{t*}Q_j^t$ for $m = 1,\ldots,M$ and $t = 0, 1, \ldots, T$ and $L(x, y)$ is the logarithmic mean defined by (62).

All we have done is establish the exact index counterpart to (49), assuming that the functional form for the annual aggregator function h has the more restrictive CES functional form defined by (71) and (67), instead of a general flexible functional form as in section 6. However, with the special structure of intertemporal preferences defined by (66) and (67), the equalities (70) and (72) established for calendar years can be extended to noncalendar years; i.e., to any consecutive run of M seasons. For example, we can establish the following counterparts to (70) and (48) for $t = 0, 1, \ldots, T-1$:

$$\begin{aligned} &\psi^{-1}\{\Sigma_{m=2}^{M}\beta_m\psi[f^m(q^{tm})] + \beta_1\psi[f^1(q^{t+1,1})]\} \\ &\quad = max_{x^1,\ldots,x^M}\{\psi^{-1}(\Sigma_{m=2}^{M}\beta_m\psi[f^m(x^m)] + \beta_1\psi[f^1(x^1)]) : \\ &\qquad\qquad \Sigma_{m=2}^{M}p^{tm*}\cdot x^m + p^{t+1,1*}\cdot x^1 = \Sigma_{m=2}^{M}p^{tm*}\cdot q^{tm} + p^{t+1,1*}\cdot q^{t+1,1}\} \\ &\quad = \psi^{-1}\{\Sigma_{m=2}^{M}\beta_m\psi[Q_m^t] + \beta_1\psi[Q_1^{t+1}]\} \\ &\quad = max_{Q_1,\ldots,Q_M}\{\psi^{-1}[\Sigma_{m=2}^{M}\beta_m\psi(Q_m) + \beta_1\psi(Q_1)] : \\ &\qquad\qquad \Sigma_{m=2}^{M}P_m^{t*}Q_m + P_1^{t+1*}Q_1 = \Sigma_{m=2}^{M}P_m^{t*}Q_m^t + P_1^{t+1*}Q_1^{t+1}\} \end{aligned} \tag{73}$$

where the P_m^{t*} and Q_m^t are defined by (44) - (47) with the p^{tm*} now defined by (69) instead of (16). The moving year utility maximization problems in (73) have dropped the quantities of season 1 in t and added those of season 1 in $t+1$. Equations (70) when $t = 0$ can be combined with (73) and the fact that the Sato-Vartia quantity index Q_{SV} is exact for the h defined by (71) and (67) to yield the following exact relationships for $t = 0, 1, \ldots, T-1$:

$$\begin{aligned} &\psi^{-1}\{\beta_1\psi[f^1(q^{t+1,1})] + \Sigma_{m=2}^{M}\beta_m\psi[f^m(q^{tm})]\}/\psi^{-1}\{\Sigma_{m=1}^{M}\beta_m\psi[f^m(q^{0m})]\} \\ &\quad = \psi^{-1}\{\beta_1\psi(Q_1^{t+1}) + \Sigma_{m=2}^{M}\beta_m\psi(Q_m^t)\}/\psi^{-1}\{\Sigma_{m=1}^{M}\beta_m\psi(Q_m^0)\} \\ &\quad = Q_{SV}(P_1^{0*},\ldots,P_M^{0*};P_1^{t+1*},P_2^{t*},\ldots,P_M^{t*};Q_1^0,\ldots,Q_M^0;Q_1^{t+1},Q_2^t,\ldots,Q_M^t). \end{aligned} \tag{74}$$

In evaluating the Sato-Vartia quantity index on the right hand side of (74), we use the base year aggregate discounted seasonal prices $P_1^{0*},\ldots,P_M^{0*}$, the base year seasonal aggregates $Q_1^0,\ldots,Q_M^0$, the year $t+1$ aggregate season 1 discounted price P_1^{t+1*} followed by the year t season 2 to M discounted prices $P_2^{t*}, \ldots, P_M^{t*}$ and the year $t+1$ season 1 quantity aggregate Q_1^{t+1} followed by the year t season 2 to M quantity aggregates $Q_2^t, \ldots, Q_M^t$.

In a similar fashion, the aggregate seasonal price and quantity data constructed using (44) - (47) for any run of M consecutive seasons, can be rearranged and inserted into the Sato-Vartia index number formula, and the resulting number times the (discounted) value of base year consumption, $\Sigma_{j=1}^M p^{0j*} \cdot q^{0j} = \Sigma_{j=1}^M P_j^{0*} Q_j^0$,

$$\begin{aligned} Q_{tm} \equiv Q_{SV}(&P_1^{0*}, \ldots, P_M^{0*}; P_1^{t+1*}, \ldots, P_{m-1}^{t+1*}, P_m^{t*}, \ldots, P_M^{t*}; \\ &Q_1^0, \ldots, Q_M^0; Q_1^{t+1}, \ldots, Q_{m-1}^{t+1}, Q_m^t, \ldots, Q_M^t)\Sigma_{j=1}^M p^{0j*} \cdot q^{0j} \end{aligned} \quad (75)$$

is an estimate of the consumer's real consumption in the moving year starting in season m of year t expressed in constant dollars pertaining to the beginning of calendar year 0.

We can divide the quantity index Q_{tm} into the discounted value ratio of the moving year starting in season m of year t to the base year to obtain a price index P_{tm}:

$$P_{tm} \equiv [\Sigma_{i=m}^M p^{ti*} \cdot q^{ti} + \Sigma_{j=1}^{m-1} p^{t+1,j*} \cdot q^{t+1,j}]/[\Sigma_{i=1}^M p^{0i*} \cdot q^{0i}]Q_{tm}. \quad (76)$$

Due to the fact that discounted price vectors p^{tm*} appear in (75) and (76) instead of the nominal price vectors p^{tm}, it is difficult to interpret the moving year price index P_{tm} that is defined by (76), just as it was difficult to interpret our earlier calendar year price indexes defined by (18). However, our focus here is on the moving year quantity indexes Q_{tm} defined by (75). The main advantage of these over the single and two stage calendar year quantity indexes defined earlier by (17) and (49) is their *timeliness*: at the end of *each* season of each year, a moving year quantity index can be calculated that will enable economic policy makers to accurately determine the progress of the economy over the current noncalendar year compared to the base calendar year. A second advantage is that they are *comprehensive*; i.e., they include *all* of the seasonal commodities whereas the short term season to season quantity indexes defined in the previous section by (54) were also timely but they had to exclude most seasonal commodities. A third advantage is that they *do not have to be seasonally adjusted*, since the quantities pertaining to an entire year starting at season m of year t are compared to the quantities pertaining to a base year. Thus the moving year quantity indexes Q_{tm} defined by (75) can be viewed as seasonally adjusted constant dollar consumption series at annual rates and the analysis in this section provides a rigorous justification for the use of these series from the viewpoint of the economic approach.

A in section 6, we recommend that the seasonal aggregates Q_m^t and P_m^{t*} be defined using Fisher ideal indexes for the seasonal bilateral indexes Q^m and P^m that appeared in (46) and (47). Of course, Statistical Agencies may have to approximate these Fisher indexes by Paasche and Laspeyres indexes and it may also be necessary to approximate the Sato-Vartia quantity and price indexes in (75) and (76) by Paasche and Laspeyres indexes as well. Provided that the base year is changed fairly frequently, these first order approximations should be adequate. In low inflation contexts (i.e., less than 5% per year), it may also be possible to approximate adequately the moving year quantity indexes Q_{tm} defined by (75) by replacing the discounted price vectors p^{tm*} defined by (69) by the nominal price vectors p^{tm}; this replacement will also occur in (44) -(47). Replacing discounted by nominal prices in (76) means that the resulting moving year price index P_{tm} can be regarded as a normal (seasonally adjusted) annual price index. Making these Paasche and Laspeyres approximations and using nominal prices p^{tm} in place of the discounted prices p^{tm*} causes (76) to become the "indice sensible" that was used as a seasonally adjusted consumer price index by the French Statistical Agency INSEE (1976; 67-68) for several years. Diewert (1983c; 1040), using Turvey's (1979) artificial data on seasonal consumption, also calculated some approximations to the moving year price indexes defined by (76): Diewert used Turvey's nominal prices instead of discounted prices and compared the results of using Laspeyres, Paasche, Fisher ideal and translog or Törnqvist (1936) price indexes in both stages of the aggregation. The choice of index number formula did not matter very much for that data set.[71]

In the following section, we regard (75) as an index number method of seasonal adjustment and compare this method with more traditional statistical methods of seasonal adjustment.

9. Econometric Versus Index Number Methods of Seasonal Adjustment

What we have done in the previous section is to show that if we use the Sato-Vartia quantity index, Q_{SV} defined by (75), to aggregate up the year over year seasonal indexes, then we can make exact index number comparisons for any consecutive string of M seasons with the base year. These moving year indexes have no seasonal components and hence can be regarded as seasonally adjusted "monthly" series at annual rates.

Instead of using the Sato-Vartia index Q_{SV} in (75), a superlative quantity index such as the Fisher ideal Q_F could be used to approximate Q_{SV}.[61] If the moving year is a calendar year, then the resulting $Q_F(P_1^{0*},\ldots,P_M^{0*};P_1^{t*},\ldots,P_M^{t*};\,Q_1^0,\ldots,Q_M^0;Q_1^t,\ldots,Q_M^t)$ reduces to our preferred two stage annual index defined by (49), where $Q^a \equiv Q_F$. In the general case where the Fisher quantity index is defined for the moving year starting at season m of year t, $Q_F \equiv [Q_P Q_L]^{1/2}$ where the Paasche and Laspeyres quantity indexes, Q_P and Q_L, are evaluated at the same aggregate seasonal prices and quantities and can be regarded as share weighted moving averages of the moving year seasonal quantity aggregates.

As a further refinement, we can "center" the series of moving year quantity indexes $Q_{t,m}$ defined by (75). If we have monthly data so that the number of seasons M equals 12, then $Q_{t,m}$ represents the aggregate quantity of a moving year starting at month m of t relative to the aggregate quantity of a base year. An estimate of the annual quantity *centered* around month m of year t compared to the quantity of the base year is

$$Q^c_{t,m} \equiv \begin{cases} (1/2)Q_{t,m-6} + (1/2)Q_{t,m-5}; & t = 0,1,\ldots,T-1; \quad m = 7,\ldots,12 \\ (1/2)Q_{t-1,m+6} + (1/2)Q_{t-1,m+7}; & t = 1,2,\ldots,T; \quad m = 1,\ldots,5 \\ (1/2)Q_{t-1,12} + (1/2)Q_{t,1}; & t = 1,\ldots,T; \quad m = 6. \end{cases} \tag{77}$$

We cannot provide centered monthly quantity estimates for the first and last 6 months; i.e., $Q^c_{t,m}$ is not defined for $t = 0$ and $m = 1, 2, \ldots, 6$ and for $t = T$ and $m = 7, 8, \ldots, 12$.

Our suggested index number method of seasonal adjustment is not really a seasonal adjustment method.[62] Our index numbers Q_{tm} defined by (75) simply compare a moving year aggregate to a corresponding base year aggregate. *Thus we have changed the question that we are trying to answer.* The centered index number comparisons Q^c_{tm} of the form (77) are averages of the more fundamental comparisons made in (75), where the averaging is done so that the resulting centered estimates will more closely resemble a conventional seasonally adjusted series at annual rates.

In Appendix 1 below, we compare official U.S. seasonally adjusted at annual rates data on quarterly GDP over the years 1959-1988 with moving year centered index numbers which aggregate the quarterly unadjusted data published in the Bureau of Economic Analysis [1992].[63] We found that our suggested index number method for seasonal adjustment performs as well as the official X-11 method. The turning points are basically the same. The main differences are: (i) the index number adjusted series is smoother and (ii) the X-11 adjusted series grows more slowly.[64] The reason for the second difference is that the X-11 series is constructed by seasonally adjusting the U.S. fixed base quarterly (unadjusted) quantity series whereas the unadjusted quarterly chain data is the input into the index number formula.[65] Our results are consistent with the fixed 1987 base year Laspeyres and chained comparisons of U.S. real GDP over the years 1959-1987 made by Young (1992; 36), who found that the average rate of growth of the fixed base GDP index numbers was 3.1% compared to 3.4% per year for the chain indexes.[66] Users of U.S. seasonally adjusted data should be made aware that it is fixed base data that is being seasonally adjusted. When the base year is changed, fairly substantial changes in growth rates can occur in the official seasonally adjusted fixed base data.

The performance results in Appendix 1 are significant because the index number method offers a number of advantages over the X-11 method: (i) The index number method can be explained fairly simply.[67] (ii) There are many significant unannounced choices that must be made by the statistican-operator of the X-11 method (i.e., multiplicative or additive seasonals, treatment of outliers, etc.), whereas the index number method involves only two easily stated choices.[68] (iii) Final seasonal adjustment factors using the X-11 method are not available until 2 or 3 years of additional unadjusted data become available, whereas, indexes of the form (75) will be available almost immediately after the last season in the moving year and the centered indexes of the form (77) will be available after an additional 6 months. The Statistical Agency will avoid the current embarrassing problem of trying to explain why the seasonally adjusted series are still being revised years after the preliminary series are released. (iv) The index number method of aggregation simultaneously seasonally adjusts (normalized) prices and quantities (recall (75) and (76) above) whereas statistical methods of seasonal adjustment separately adjust prices, quantities and values without respecting the fact that only 2 of these 3 variables are independent. (v) Finally, statistical seasonal adjustment methods that allow for changing seasonals run into a severe identification problem and the resulting seasonal factors that these statistical methods churn out are not well defined from a theoretical point of view.[69]

The econometric methods do have the advantage that they can be applied in situations where there is quantity but not price information, i.e., the X-11 method can seasonally adjust an unemployment series but an index number method cannot.

It should be emphasized that the moving year quantity indexes defined by (75) are sufficient statistics for defining the centered moving year quantity indexes defined by (77). Thus the Statistical Agency should strive to provide moving year quantity and price indexes of the form (75) and (76) on a timely basis: users can easily perform the simple arithmetic operations inherent in forming the centered moving year indexes of the form (77).

Our specific assumptions on intertemporal preferences represented by (66) and (67) led to the specific Sato-Vartia exact index number formula (75) where the "monthly" aggregates P_m^{t*} and Q_m^t were formed using superlative index number formulae in (44) - (47). Since in many situations, it may be necessary to approximate both the "monthly" price indexes P^m which appear in (47) and the Sato-Vartia price index P_{SV} which appears in (76) by Laspeyres price indexes. Then the corresponding quantity indexes in (46) and (75) will be Paasche quantity indexes. These Paasche and Laspeyres indexes will approximate their superlative and Sato-Vartia counterparts will be acceptable approximations, provided that the base year is changed fairly frequently.

Another approximation to our recommended theoretically exact indexes defined by (44) - (47) and (75) - (76) occurs if the inflation adjusted prices p^{tm*} defined by (69) and used in (44) - (47) are replaced by the corresponding unadjusted spot price vectors p^{tm}. This will make little difference to the moving year quantity indexes defined by (75) and (77) provided that: (i) inflation is "low" and (ii) seasonal fluctuations are not "too" erratic. Numerical experiments will be required before we can be more precise.

10. Conclusion

We have discussed the problem Statistical Agencies face when constructing price and quantity aggregates under conditions of high inflation when there are seasonal commodities. Without seasonal commodities, the index number problem is still straightforward (but expensive): the Statistical Agency must collect subannual price and quantity (or value) information more frequently in order to make the subannual periods of time short enough so that variations in prices within the periods can be neglected.[70] However, when there are seasonal commodities, this solution to the high inflation index number problem is not valid: we cannot make meaningful bilateral index number comparisons (from the viewpoint of the economic approach) between consecutive months or quarters if the dimensionality of the commodity space varies from period to period.

The assumptions on preferences that we have made provide justifications for three types of seasonal index number comparisons that Statistical Agencies should provide:

(i) For measuring short term price change, the approach outlined in section 7 should be used; i.e., a season to season short run price index using only nonseasonal (and type (a) seasonal) commodities should be constructed. These short term indexes would be used as deflators[71] when constructing the annual quantity indexes in (iii).

(ii) The year over year seasonal indexes defined by (39) and (40) in section 5 should also be constructed. The assumptions on preferences required to justify these are the least restrictive. The business community may find these indexes the most useful.

(iii) Finally, the moving year price and quantity indexes defined by (75) and (76) in section 9 should also be calculated.[72] These indexes will serve as seasonally adjusted price and quantity indexes (at annual rates). If there is low inflation, spot prices p^{tm} can be used in place of the normalized prices p^{tm*} in (45) - (48) and (75) - (76).

For each of the above three indexes, the Statistical Agency will have to decide whether to provide Paasche and Laspeyres or superlative versions. From the viewpoint of economic theory, the superlative versions are better but they will be more costly and less timely. In the long run, Statistical Agencies will be able to make use of electronically recorded data on the sales of commodities to produce timely superlative indexes.[73] However, in the short run, difficult choices must be made on how to produce price and quantity indexes when there are seasonal commodities and high inflation.

Appendix 1: U.S. Seasonally Adjusted and Centered Moving Year Estimates

The raw data for our comparisons are from the Bureau of Economic Analysis (1992): seasonally unadjusted estimates of U.S. GDP from quarter 1 of 1959 to quarter 4 of 1988 (120 quarters in all) are from Table 9.1; implicit price deflators for GDP using chain type weights are from Table 7.2 and estimates of quarterly GDP seasonally adjusted at annual rates are from Table 1.2. The price index was normalized to equal 1 in the third quarter of 1987. Dividing the seasonally unadjusted GDP by the price index gave us the series $Y_t,\ t = 1, \ldots, 120$ (note the changed notation here, with t denoting consecutive quarters). The series Y_t is plotted in Figure 1. The units are 100 millions of 1987 third quarter dollars. The series of 4th quarter observations are indicated by the sharp peaks joined up by dashed lines. The seasonal fluctuations are evolving over time.

Denote the Fisher ideal fixed base moving year index by Q_t, where t indicates the first quarter of the moving year and the base year consists of the 4 quarters of 1987.

(We made no adjustment for general inflation since it was not high). Q_t is defined for $t = 1, 2, \ldots, 117$. and the centered index is given by

$$Q_t^c \equiv (1/2)Q_{t-1} + (1/2)Q_{t-2}; \quad t = 3, 4, \ldots, 118. \qquad (A1)$$

These centered indexes are plotted as the solid line in Figure 2 (and denoted by QF and measured again in 100 millions of 1987 dollars). The official seasonally adjusted U.S. constant dollar GDP series for the same 116 quarters is also plotted as the dashed line in Figure 2 (and denoted by SAY). SAY grows more slowly and is more erratic than QF but both series have roughly the same turning points and hence both can serve as guides business cycle movements.

Figure 1

Figure 2

Footnotes

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1. See Middleditch (1918), Paton (1920; 2), Sweeney (1927) (1928) (1964) and Baxter (1975; 17-35) (1984; 38-57).

2. This classification corresponds to Balk's (1980a; 7) (1980b; 110) (1980c; 68) narrow and wide sense seasonal commodities.

3. This classification is due to Mitchell (1927; 236). See Mitchell (1927; 237) and Granger (1978; 33) examples of seasonal fluctuations due to custom.

4. This is not quite true: both Diewert (1980; 506-508) and Balk (1981) proposed procedures which will work over commodity spaces of varying dimensions. See section 7.

5. References on the economic approach include Konüs (1924), Pollak (1989), Samuelson and Swamy (1974) and Diewert (1976) (1981) (1983a) (1983b) (1993a).

6. Using the nonparametric tests for maximizing behavior due to Afriat (1967) and Diewert (1973; 424), we can test whether a given set of price and quantity data are consistent with the maximization of a homothetic or linearly homogeneous utility function;

see Diewert (1981; 198-199). If a combination of seasonal and nonseasonal data pass this test, then the seasonal commodities are of type (ii) (a).

7. See Diewert(1980; 506-508) (1983c) on the economic approach to seasonal indexes. This paper focuses on the theory of the seasonal consumer price index. An analogous theory exists for the seasonal producer price index with separability assumptions on the producer's intertemporal production function or factor requirements function. See Fisher and Shell (1972) and Diewert (1980) (1983b).

8. The role of separability assumptions in the economic approach to index number theory is laid out in chapters 2 and 3 of Pollak (1989) and chapter 9 of Blackorby, Primont and Russell (1978). The latter book provides a general exposition of separability.

9. The present paper generalizes Diewert's (1983c) earlier economic approach.

10. Our single consumer theory can be extended to groups of consumers using Pollak's (1981; 328) Social Cost of Living Index; see Diewert (1983a; 190-192) (1993b; 294). We do not deal with sampling problems in the present paper.

11. For example, should indirect taxes be included in the consumer price index? The structure of the relevant utility maximization problem gives us guidance on this issue. This advantage of the economic approach has been stressed by Jack Triplett over the years.

12. Some precursors of Mudgett and Stone in recognizing that seasonal commodities should be distinguished as separate commodities in the seasons that they are available

were Marshall (1887; 373) and Bean and Stine (1934; 34).

13. f is defined over the annual commodity space of dimension $\Sigma_{m=1}^{M} N_m \equiv N^*$; i.e., each "physical" commodity in each season is treated as a separate economic commodity from the perspective of the annual utility f. The concavity assumption on f can be replaced by the weaker condition of continuity; see Diewert (1974; 111).

14. See Diewert (1976; 116) (1981; 180-193) (1983a; 184) (1993a; 45-50).

15. See Diewert (1976; 116) for the history of this result.

16. See Diewert (1976; 117) for this terminology. Diewert (1976; 137-138) regarded Q_F as the best superlative index number formula since it is exact for both Leontief (no substitution) and linear (perfect substitutability) aggregator functions and it is the only superlative index that is consistent with revealed preference theory. Diewert (1992; 221) also showed that Q_F had good axiomatic properties.

17. The terminology is due to Samuelson and Swamy (1974; 572).

18. This relationship was first noticed by Fisher (1896; 75). Fisher (1896; 13 and 69) also defined the expected real interest rate $\bar{r}_{tm}$ in terms of the nominal interest rate r_{tm} and an expected commodity inflation rate i_{tm} going from season m to season $m+1$ of year t as follows: $(1 + r_{tm}) = (1 + \bar{r}_{tm})(1 + i_{tm})$.

19. This type of deflation was used by German accountants to stabilize (or make comparable) accounting values pertaining to different time periods during the German hyper-

inflation of 1923; see Sweeney (1927) (1928).

20. This type of reevaluation to make accounting values comparable is also used by accountants in high inflation countries; e.g., see Wasserman (1931; 10).

21. Since it is difficult to distinguish type (ii) (a) from type (ii) (b) seasonal commodities, it may be more practical to stick with nonseasonal ones. Of course, there may be difficulties in distinguishing nonseasonal from seasonal commodities as well.

22. This commodity basket approach to deflating the value of money to make it comparable over time dates back at least to William Fleetwood who wrote in 1707; see Ferger (1946). The use of the cost of a basket of goods as an index number was extensively developed by Lowe (1823; 331-346) and Scrope (1833; 401-425) who applied this idea to many practical problems of indexation. This commodity standard idea for adjusting the value of money has been independently discovered many times; see for example Marshall (1887; 371), Fisher (1911; ch. 13) and the many references in Fisher (1920; 291-294).

23. This point was first made by Walsh (1901; 96) (1921;88) and Davies (1924; 183) (1932;59). For more recent discussions on unit values, see Dalen (1992; 135), Diewert (1995; 20-24) and Balk (1995a).

24. See the data on the German hyperinflation in Sweeney (1927; 182).

25. Our observation here is substantially due to Marshall (1887; 374).

26. Assumption (27) is restrictive: it says that the consumer is indifferent to the annual consumption of each commodity taking place in a single season or spread out across seasons.

27. Note that these conditions are Hicksian (1946; 312) aggregation conditions which guarantee the existence of annual aggregates. In fact, if conditions (28) hold, we do not have to make the restrictive assumption (27) in order to determine that $[q^{t1}, q^{t2}, \ldots, q^{tM}]$ solves (15). To determine the annual aggregator function g^* under conditions (28), let $c(p^{t1*}, \ldots p^{tM*})$ be the unit cost function dual to $f(x^1, \ldots, x^M)$. Define the N variable unit cost function $c^*(p^t) \equiv c(p^t, \ldots, p^t)$. Then $g^*(\Sigma_{m=1}^M x^m)$ is dual to c^*.

28. Satisfaction of (33) also ensures equality of the two Laspeyres indexes, $Q_L^*(P^{s*}, P^{t*}, Q^s, Q^t) \equiv P^{s*} \cdot Q^t / P^{s*} \cdot Q^s$ and $Q_L(p^{s*}, p^{t*}, q^s, q^t) \equiv p^{s*} \cdot q^t / p^{s*} \cdot q^s$ and the Paasche indexes, $Q_P^*(P^{s*}, P^{t*}, Q^s, Q^t) \equiv P^{t*} \cdot Q^t / P^{t*} \cdot Q^s$ and $P_P(p^{s*}, p^{t*}, q^s, q^t) \equiv p^{t*} \cdot q^t / p^{t*} \cdot q^s$.

29. For example see Bean and Stine's (1924; 31) Type D index number or Rothwell (1958; 70). Incidentally, Flux (1921; 185) also proposed (and used) Bean and Stine's Type B index and Crump, in his discussion of Flux's (1921; 207) paper, proposed Bean and Stine's Type A index number. Finally, Bean and Stine's (1924; 31) Type C index number has come to be known as the Rothwell (1958; 71) index.

30. Actually, the exact index number approach used in this section in conjunction with the homogeneous separability assumptions (36) can be viewed as an extension of Shephard's (1953; 64-71) aggregation theory; see also Diewert (1974; 151).

31. Multiply both sides of the constraint in (37) by the discount factor ρ_{tm} and the resulting constraint becomes $p^{tm*} \cdot x^m = p^{tm*} \cdot q^{tm}$. Thus the nominal price vectors p^{sm} and p^{tm} in (39) and (40) can be replaced by the normalized price vectors p^{sm*} and p^{tm*}.

32. This requires that h be increasing. This technique was used by Shephard (1953; 64-71) (1970; 114-123) and Diewert (1974; 164-165) under various regularity conditions.

33. If there are no type (i) seasonal commodities, then consider an alternative two stage procedure where for each "physical" commodity, we aggregate over seasons within a year in the first stage and then aggregate over these "annual" commodities in the second stage, using say Laspeyres price indexes and Paasche quantity indexes at each stage. This procedure would give the same answer as the single stage procedure (41) if the Q in (41) were the Paasche index $Q_P(p^{0*}, p^{t*}, q^0, q^t) \equiv p^{t*} \cdot q^t / p^{t*} \cdot q^0$. These alternative two stage aggregation procedures were considered by Balk (1980a; 25) and Diewert (1980; 506-508). Different separability assumptions are required to justify each procedure.

34. In this case, equation (36) will hold even though Leontief aggregator functions are not strictly increasing in all arguments. If h is Leontief, then $h(Q_1, \ldots, Q_M) \equiv min_m\{Q_m/b_m : m = 1, \ldots, M\}$ where the b_m are positive constants. The unit cost function is $\Sigma_{m=1}^{M} b_m P_m$.

35. The Leontief functional form is not flexible; i.e., its dual unit cost function can provide only a first order approximation to an arbitrary differentiable unit cost function.

36. Diewert's (1978; 888) results required that the second order approximations be taken around a point where the period t price and quantity vectors equal the period 0 price and quantity vectors. However, the same second order approximation result will hold if these equality restrictions are relaxed to proportionality restrictions, since superlative indexes are homogeneous of degree 0 or 1 in their price and quantity vector arguments.

37. Zarnowitz (1961; 246) seemed to feel that it is possible to somehow estimate cardinally comparable seasonal subutility functions f^m. Balk (1980a; 21) denied this.

38. We want to include all nonseasonal and type (ii) (a) seasonal commodities in the $\tilde{x}^{tm}$ vectors to make the coverage of the resulting "monthly" price indexes as broad as possible. However, we must exclude type (i) and type (ii) (b) seasonal commodities from the "monthly" aggregator function ϕ, since inclusion of these would cause the resulting ϕ to shift as climate and customs changed across the seasons, making "monthly" comparisons impossible. In practice, it will be difficult to decide what is a type (ii) (a) seasonal commodity.

39. We also use the positivity of the discount factors δ_t and ρ_{tm} in deriving (53).

40. Rothwell (1958; 71) noted that the problem of making price comparisons between seasons with different market baskets is formally identical to the problem of making international comparisons between countries with different market baskets. This suggests that the symmetric methods used in making international comparisons could be applied to the

problem of aggregating up the many bilateral price comparisons in (55) into a consistent sequence of "monthly" price levels. Balk (1981; 74) in fact implemented this idea, calculating a system of EKS (see Gini (1931; 12), Eltetö and Köves (1964) and Szulc (1964)) monthly purchasing power parities for Dutch fruit and vegetables. However, Walsh (1901; 399) and Balk (1981; 77) also noted a practical disadvantage to the use of these symmetric methods: the price levels have to be recalculated each time a new observation is added.

41. See the discussion and references in Diewert (1993a; 52-55).

42. In fact, it was these problems that led Julius Lehr (1885; 45-46) and Alfred Marshall (1887; 373) to introduce the chain system.

43. Our formal model needs to be modified to deal with the problem of new commodities. Mudgett (1951; 46) called the error in an index number comparison that was introduced by the use of the highest common factor method the homogeneity error.

44. Using annual Canadian data for 13 categories of consumption over the years 1947 to 1971, Diewert (1978; 894) provided some evidence to support these theoretical approximation results.

45. The chain index will give the correct answer if $\tilde{P}$ satisfies Walsh's (1901; 389) (1924; 506) multiperiod identity test; see Diewert (1993a; 40) for a discussion of this test. However, the Paasche, Laspeyres and all known superlative indexes do not satisfy this test

and thus are vulnerable to the Szulc-Hill criticism of the chain system. Incidentally, Walsh (1901; 401) (1924; 506) was the first to make this criticism.

46. The reason for this statement is that Fisher (1922; 280-283) (and others) have found that $\tilde{P}_F$ satisfies Walsh's multiperiod identity test to a high degree of approximation.

47. See also Hofsten (1952; 97) and Fisher and Shell (1972; 101)

48. Diewert (1980; 502-503) suggested an econometric approach to the estimation of reservation prices but did not implement it. Hausman (1997) seems to have been the first to implement such an econometric approach.

49. As mentioned earlier in section 6, the Vartia I (1974; 66-67) (1976) price index was used to establish the approximate consistency in aggregation of superlative indexes. It is interesting to note that Montgomery (1937; 37) defined the Vartia I index much earlier and also established its consistency in aggregation properties; see Montgomery (1937; 40-48).

50. For the properties of the logarithmic mean and references to the mathematics literature, see Carlson (1972).

51. Lau (1979; 75-81) clarified and extended the class of functions that P_{SV} is exact for. The CES aggregator function is equal to a positive constant times a weighted mean of order r. For the properties and axiomatic characterizations of means of order r, see Hardy, Littlewood and Polya (1934; 12-19) and Diewert (1993c; 381).

52. Reinsdorf and Dorfman (1995) show that P_{SV} fails to satisfy the four monotonicity axioms listed in Diewert (1992; 220) that are due to Eichhorn and Voeller (1976; 23) and Vogt (1980; 70). The Fisher index P_F satisfies these axioms.

53. P_{SV} is not pseudosuperlative (see Diewert (1978; 888)) either; i.e., when we evaluate the first and second order partial derivatives of $P_{SV}(p^0, p^1, q^0, q^1)$ around an equal price ($p^0 = p^1$) and equal quantity ($q^0 = q^1$) point, we find that the first order derivatives of P_{SV} coincide with the corresponding first order derivatives of a superlative index tabled in Diewert (1978; 893) but the second order derivatives do not. This is what we would expect since P_{SV} is exact for CES and two stage mixtures of CES and Cobb-Douglas functions (see Lau (1979; 75-81) for the precise results) and these functions can provide only first order approximations to arbitrary N commodity aggregator functions.

54. Actually, we are describing only the first stage in Balk's (1981; 73) procedure. In the second stage of his procedure, Balk (1981; 74) uses the multilateral Gini (1931; 12), Eltetö and Köves (1964) (EKS) index and Szulc (1964) (EKS) index to eliminate the influence of a base period on his seasonal price indexes.

55. Since seasonal price and quantity changes can be huge, the choice of the index number formula makes a large difference. When Balk (1980a; 41) compared his Sato-Vartia indexes for Dutch fruit and vegetables with an alternative index number formula, he found some differences in the 50% range. Also Reinsdorf and Dorfman (1995; table

1) found substantial differences between the Sato-Vartia price index and the superlative Fisher and Törnqvist (1936) price indexes for some artificial data

56. For a discussion of the problems involved in constructing user costs for consumer durables and references to the literature, see Diewert (1983a; 211-216).

57. The term “moving year” is due to Mendershausen (1937; 245). Diewert (1983c; 1029) earlier used the term “split year” comparison to describe a variable year end index number comparison. Following the terminology used by Crump (1924; 185) in a slightly different context, we could also use the term “rolling year” comparison.

58. Diewert (1983c; 1034) assumed that U was the simple sum of seasonal utilities, $\Sigma_{t=0}^{T}\Sigma_{m=1}^{M} f^m(x^{tm})$. This is a special case of (66) and (67) with the $\beta_m = 1$ and $\alpha = 1$.

59. Recall that $\psi(z) \equiv z^\alpha$ if $\alpha \neq 0$. If $\alpha < 0$ then the max in (68) is replaced by a min, but equations (70) are still satisfied.

60. Once the Q_{tm} or P_{tm} have been defined by (75) or (76) for $t = 0$ and $m = 1, \ldots, M$, the chain principle can be used to relate the prices and quantities of each moving year to those of the preceding moving year; see Diewert (1983c; 1031-1032) for comparisons of fixed base and chained moving year price indexes using the Turvey (1979) data.

61. Since superlative indexes are exact for flexible aggregators, the flexible aggregator function can approximate the CES aggregator function in (74) to the second order.

62. For material on time series methods of seasonal adjustment, see Bell and Hillmer (1984), Hylleberg (1992) and Findley (1997).

63. Instead of the Sato-Vartia quantity index, we used the Fisher ideal quantity index in (75). We did not deflate the quarterly prices by an index of purchasing power since inflation was "small" over this period.

64. The average quarterly rate of growth for the official X-11 adjusted series was .78% compared to .85% per quarter for our centered Fisher ideal moving year series.

65. Until recently, most U.S. long term quantity series were constructed using a fixed base Laspeyres quantity index so that additivity of components could be preserved. With the recent huge increases in the quantity of computers and their equally huge declines in price, the use of fixed base indexes has become unworkable: changing the base year leads to dramatic revisions in economic history. This illustrates a point of Hill's (1988) (1993): the base period in fixed base index numbers must be changed reasonably frequently.

66. This difference in annual growth rates is .3% per year which is approximately equal to four times our quarterly difference in growth rates of .07% per quarter.

67. See William R. Bell and Steven C. Hillmer (1984; 291).

68. The two choices are variants of (75): (i) should the inflation adjusted normalized prices p^{tm*} defined by (69) be replaced by the unadjusted spot prices p^{tm} and (ii) should

the Sato-Vartia index number formula Q_{SV} which appears in (75) be replaced by some other index number formula?

69. See the discussion by Anderson (1927; 552-553).

70. Diewert (1995; 22) advocated this solution to the index number problem under high inflation but he neglected the seasonal commodities problem.

71. Recall equations (22) and (69).

72. Under conditions of high inflation, the price indexes defined by (76) will be difficult to interpret and hence the Statistical Agency would not have to report them. The primary focus should be on the production of the moving year quantity indexes defined by (75).

73. Mr. William Hawkes informs me that the A.C. Nielsen company based in the U.S. distinguished 1.65 million separate product codes as of September, 1995; i.e., this company has detailed price and quantity information by region on all of these commodities.

References

Afriat, S.N. (1967), "The Construction of Utility Functions from Expenditure Data", *International Economic Review* 8, 67-77.

Anderson, O. (1927), "On the Logic of the Decomposition of Statistical Series into Separate Components", *Journal of the Royal Statistical Society* 90, 548-569.

Archibald, R.B. (1977), "On the Theory of Industrial Price Measurement: Output Price Indexes", *Annals of Economic and Social Measurement* 6, 57-72.

Balk, B.M. (1980a), *Seasonal Products in Agriculture and Horticulture and Methods for Computing Price Indices*, Statistical Studies no. 24, The Hague: Netherlands Central Bureau of Statistics.

Balk, B.M. (1980b), "Seasonal Commodities and the Construction of Annual and Monthly Price Indexes", *Statistische Hefte* 21:2, 110-116.

Balk, B.M. (1980c), "A Method for Constructing Price Indices for Seasonal Commodities", *Journal of the Royal Statistical Society* A, 143, 68-75.

Balk, B.M. (1981), "A Simple Method for Constructing Price Indices for Seasonal Commodities", *Statistische Hefte* 22:1, 72-78.

Balk, B.M. (1995a), "On the Use of Unit Value Indices as Consumer Price Subindices", March 14, Department of Statistical Methods, Statistics Netherlands, P.O. Box 4000, 2270 JM Voorburg, The Netherlands.

Balk, B.M. (1995b), "Axiomatic Price Index Theory: A Survey", *International Statistical Review* 63, 69-93.

Baxter, W.T. (1975), *Accounting Values and Inflation*, London: McGraw-Hill.

Baxter, W.T. (1984), *Inflation Accounting*, Oxford: Philip Allan Publishers.

Bean, L.H. and O.C. Stine (1924), "Four Types of Index Numbers of Farm Prices", *Journal of the American Statistical Association* 19, 30-35.

Bell, W.R. and S.C. Hillmer (1984), "Issues Involved with the Seasonal Adjustment of Economic Time Series", *Journal of Business and Economic Statistics* 2, 291-320.

Blackorby, C., D. Primont and R.R. Russell (1978), *Duality, Separability and Functional Structure: Theory and Economic Applications*, New York: North-Holland.

Bureau of Economic Analysis (1992), *National Income and Product Accounts of the United States*, Volume 2, 1959-88, Washington, D.C.: U.S. Government Printing Office.

Carlson, B.C. (1972), "The Logarithmic Mean", *The American Mathematical Monthly* 79, 615-618.

Crump, N. (1924), "The Interrelation and Distribution of Prices and their Incidence Upon Price Stabilization", *Journal of the Royal Statistical Society* 87, 167-206.

Dalén, J. (1992), "Computing Elementary Aggregates in the Swedish Consumer Price Index", *Journal of Official Statistics* 8, 129-147.

Davies, G.R. (1924), “The Problem of a Standard Index Number Formula”, *Journal of the American Statistical Association*, 19, 180-188.

Davies, G.R. (1932), “Index Numbers in Mathematical Economics”, *Journal of the American Statistical Association*, 27, 58-64.

Diewert, W.E. (1973), “Afriat and Revealed Preference Theory”, *Review of Economic Studies* 40, 419-426.

Diewert, W.E. (1974), “Applications of Duality Theory”, pp. 106-171 in M.D. Intriligator and D.A. Kendrick (eds.), *Frontiers of Quantitative Economics*, Vol. II, Amsterdam: North-Holland.

Diewert, W.E. (1976), “Exact and Superlative Index Numbers”, *Journal of Econometrics* 4, 115-145.

Diewert, W.E. (1978), “Superlative Index Numbers and Consistency in Aggregation”, *Econometrica* 46, 883-900.

Diewert, W.E. (1980), “Aggregation Problems in the Measurement of Capital”, pp. 433-528 in *The Measurement of Capital*, Dan Usher (ed.), Chicago: University of Chicago Press.

Diewert, W.E. (1981), “The Economic Theory of Index Numbers: a Survey”, pp. 163-208 in *Essays in the Theory and Measurement of Consumer Behavior in Honour of Sir Richard Stone*, A. Deaton (ed.), London: Cambridge University Press.

Diewert, W.E. (1983a), "The Theory of the Cost-of-Living Index and the Measurement of Welfare Change", pp. 163-233 in *Price Level Measurement*, W.E. Diewert and C. Montmarquette (eds.), Ottawa: Statistics Canada.

Diewert, W.E. (1983b), "The Theory of the Output Price Index and the Measurement of Real Output Change", pp. 1049-1113 in *Price Level Measurement*, W.E. Diewert and C. Montmarquette (eds.), Ottawa: Statistics Canada.

Diewert, W.E. (1983c), "The Treatment of Seasonality in a Cost-Of-Living Index", pp. 1019-1045 in W.E. Diewert and C. Montmarquette (eds.), *Price Level Measurement*, Ottawa: Statistics Canada.

Diewert, W.E. (1992), "Fisher Ideal Output, Input and Productivity Indexes Revisited", *The Journal of Productivity Analysis* 3, 211-248.

Diewert, W.E. (1993a), "The Early History of Price Index Research", pp. 33-65 in *Essays in Index Number Theory, Volume 1*, W.E. Diewert and A.O. Nakamura (eds.), Amsterdam: North-Holland.

Diewert, W.E. (1993b), "Group Cost of Living Indexes: Approximations and Axiomatics", pp. 287-303 in *Essays in Index Number Theory*, Volume 1, W.E. Diewert and A.O. Nakamura (eds.), Amsterdam: North-Holland.

Diewert, W.E. (1993c), "Symmetric Means and Choice Under Uncertainty", pp. 355-433 in *Essays in Index Number Theory*, Volume 1, W.E. Diewert and A.O. Nakamura (eds.), Amsterdam: North-Holland.

Diewert, W.E. (1995), “Axiomatic and Economic Approaches to Elementary Price Indexes”, Discussion Paper No. 95-01, Department of Economics, The University of British Columbia, Vancouver, Canada V6T 1Z1.

Eichhorn, W. and J. Voeller (1976), *Theory of the Price Index*, Berlin: Springer-Verlag.

Eltetö, O, and P. Köves (1964), “On a Problem of Index Number Computation Relating to International Comparison”, *Statisztikai Szemle* 42, 507-518.

Feenstra, R.C. (1994), “New Product Varieties and the Measurement of International Prices”, *American Economic Review* 84, 157-177.

Ferger, W.F. (1946), “Historical Note on the Purchasing Power Concept and Index Numbers”, *Journal of the American Statistical Association* 41, 53-57.

Fisher, I. (1896), *Appreciation and Interest*, New York: MacMillan.

Fisher, I. (1911), *The Purchasing Power of Money*, London: MacMillan.

Fisher, I. (1920), *Stabilizing the Dollar*, New York, MacMillan.

Fisher, I. (1922), *The Making of Index Numbers*, Boston: Houghton Mifflen.

Fisher, I. (1930), *The Theory of Interest*, New York: MacMillan.

Fisher, F.M. and K. Shell (1972), *The Economic Theory of Price Indexes*, New York: Academic Press.

Flux, A.W. (1921), "The Measurement of Price Changes", *Journal of the Royal Statistical Society* 84, 167-199.

Gini, C. (1931), "On the Circular Test of Index Numbers", *Metron* 9:2, 3-24.

Granger, C.W.J. (1978), "Seasonality: Causation, Interpretation, and Implications", pp. 33-45 in *Seasonal Analysis of Economic Time Series*, A. Zellner (ed.), Washington, D.C.: Bureau of the Census.

Hardy, G.H., J.E. Littlewood and G. Polya (1934), *Inequalities*, London: Cambridge University Press.

Hausman, J. (1997), "Valuation of New Goods Under Perfect and Imperfect Competition", T.F. Bresnahan and R.J. Gordon (eds.), Chicago: The University of Chicago Press. MASS., April.

Hicks, J.R. (1940), "The Valuation of the Social Income", *Economica* 7, 105-140.

Hicks, J.R. (1946), *Value and Capital*, Second Edition, Oxford: Clarendon Press.

Hill, P. (1988), "Recent Developments in Index Number Theory and Practice", *O.E.C.D. Economic Studies* 10 (Spring), 123-148.

Hill, P. (1993), "Price and Volume Measures", pp. 379-406 in *System of National Accounts 1993*, Eurostat, IMF, OECD, UN and World bank, Brussels, Luxembourg, New York and Washington, D.C.

Hill, P. (1995), *Handbook on Inflation Accounting*, June draft, Paris: OECD.

Hofsten, E. von (1952), *Price Indexes and Quality Change*, London: George Allen and Unwin.

Hylleberg, S. (ed.) (1992), *Modelling Seasonality*, Oxford: Oxford University Press.

Institut National de la Statistique et des Etudes Economiques (1976), *Pour Comprendre l'Indice des Prix*, Paris: INSEE.

Keynes, J.M. (1930), *Treatise on Money,*, Vol. 1, London: MacMillan.

Konüs, A.A. (1924), "The Problem of the True Index of the Cost of Living", translated in *Econometrica* 7, (1939), 10-29.

Lau, L.J. (1979), "On Exact Index Numbers", *The Review of Economics and Statistics* 61, 73-82.

Lehr, Julius (1885), *Beiträge zur Statistik der Preise*, Frankfurt: J.D. Sauerländer.

Leontief, W. (1936), "Composite Commodities and the Problem of Index Numbers", *Econometrica* 4, 39-59.

Lowe, J. (1823), *The Present State of England in Regard to Agriculture, Trade and Finance*, Second edition, London: Longman, Hurst, Rees, Orme and Brown.

Marshall, A. (1887), "Remedies for Fluctuations of General Prices", *Contemporary Review* 51, 355-375.

Mendershausen, H. (1937), "Annual Survey of Statistical Technique: Methods of Computing and Eliminating Changing Seasonal Fluctuations", *Econometrica* 5, 234-262.

Middleditch, L. (1918), "Should Accounts Reflect the Changing Value of the Dollar?", *The Journal of Accountancy* 25, 114-120.

Mitchell, W.C. (1927), *Business Cycles*, New York: National Bureau of Economic Research.

Montgomery, J.K. (1937), *The Mathematical Problem of the Price Index*, London: P.S. King and Son.

Mudgett, B.D. (1951), *Index Numbers*, New York: John Wiley and Sons.

Mudgett, B.D. (1955), "The Measurement of Seasonal Movements in Price and Quantity Indexes", *Journal of the American Statistical Association* 50, 93-98.

Paton, W.A. (1920), "Depreciation, Appreciation and Productive Capacity", *The Journal of Accountancy* 30, 1-11.

Pollak, R.A. (1981), "The Social Cost-of-Living Index", *Journal of Public Economics* 15, 311-336.

Pollak, R.A. (1989), *The Theory of the Cost-of-Living Index*, Oxford: Oxford University Press.

Reinsdorf, M.B. and A.H. Dorfman (1995), "The Sato-Vartia Index and the Monotonicity Axiom", Washington, D.C.: US Bureau of Labor Statistics.

Rothwell, D.P. (1958), "Use of Varying Seasonal Weights in Price Index Construction", *Journal of the American Association* 53, 66-77.

Samuelson, P.A. and S. Swany (1974), “Invariant Economic Index Numbers and Canonical Duality: Survey and Synthesis”, *American Economic Review* 64, 566-593.

Sato, K. (1976), “The Ideal Log-Change Index Number”, *The Review of Economics and Statistics* 58, 223-338.

Scrope, G.P. (1833), *Principles of Political Economy*, London: Longman, Rees, Orme, Brown, Green and Longman.

Shephard, R.W. (1953), *Cost and Production Functions*, Princeton, N.J.: Princeton University Press.

Shephard, R.W. (1970), *Theory of Cost and Production Functions*, Princeton, N.J.: Princeton University Press.

Stone, R. (1956), *Quantity and Price Indexes in National Accounts*, Paris: The Organization for European Cooperation.

Sweeney, H.W. (1927), “Effects of Inflation on German Accounting”, *Journal of Accountancy* 43, 180-191.

Sweeney, H.W. (1928), “German Inflation Accounting”, *Journal of Accountancy* 45, 104-116.

Szulc, B. (1964), “Indices for Multiregional Comparisons”, *Przeglad Statystyczny (Statistical Review)* 3, 239-254.

Szulc, B.J. (1983), "Linking Price Index Numbers", pp. 537-566 in *Price Level Measurement*, W.E. Diewert and C. Montmarquette (eds.), Ottawa: Statistics Canada.

Törnqvist, L. (1936), "The Bank of Finland's Consumption Price Index", *Bank of Finland Monthly Bulletin* 10, 1-8.

Turvey, R. (1979), "The Treatment of Seasonal Items in Consumer Price Indices", *Bulletin of Labour Statistics*, 4th quarter, International Labour Office, Geneva, 13-33.

United Nations (1993), *Systems of National Accounts 1993*, New York: United Nations.

Vartia, Y.O. (1974), *Relative Changes and Economic Indices*, Licenciate Thesis, Department of Statistics, University of Helsinki.

Vartia, Y.O. (1976), "Ideal Log-Change Index Numbers", *Scandanavian Journal of Statistics* 3, 121-126.

Vogt, A. (1980), "Der Zeit und der Faktorumkehrtests als 'Finders of Tests' ", *Statistische Hefte* 21, 66-71.

Walsh, C.M. (1901), *The Measurement of General Exchange Value*, New York: MacMillan and Co.

Walsh, C.M. (1921), *The Problem of Estimation*, London: P.S. King and Son.

Wasserman, M.J. (1931), "Accounting Practice in France During the Period of Monetary Inflation (1919-1927)", *The Accounting Review* 6, 1-32.

Young, A.H. (1992), "Alternative Measures of Change in Real Output and Prices", *Survey of Current Business* 72 (April), 32-48.

Zarnowitz, V. (1961), "Index Numbers and the Seasonality of Quantities and Prices", pp. 233-304 in *The Price Statistics of the Federal Government*, G.J. Stigler (Chairman), New York: National Bureau of Economic Research.