MIDPOINT-YEAR BASKET INDEX AS A PRACTICAL APPROXIMATION TO SUPERLATIVE INDEX

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Paper presented at the Sixth Meeting of the
International Working Group on Price Indices
Canberra, Australia, 2-6 April 2001

According to Hill(1999), both the objective of measuring inflation and measuring the changes in the cost of living lead to the same kind of index formula in practice, provided that 'best practice' is followed. Furthermore, he proposed a kind of pure price index which uses the quantities in the third year intermediate between the base year and the observation year as contents in a basket instead of superlative indexes or pure price index which uses some average of the quantities in the base year and the observation year as contents in a basket.

Following his proposal, this paper presents the results of a test calculation for the 'midpoint-year basket index' defined as below using a dataset of 1995-base Japanese consumer price index. Shultz (1998) applied actually the identical formula named 'single year, mid-term basket index' to price and volume indices for final domestic demand and price index series of industrial production.

$$H = \frac{\sum q_{t/2} p_t}{\sum q_{t/2} p_0} = \frac{\sum \frac{w_{t/2}}{I_{t/2}} I_t}{\sum \frac{w_{t/2}}{I_{t/2}}} \tag{1}$$

where $w_{t/2} = p_{t/2} q_{t/2}$, $I_{t/2} = \frac{p_{t/2}}{p_0}$, $I_t = \frac{p_t}{p_0}$

0 is the base year, t is the observation year, $t/2$ is the midpoint - year

In practice, the 'midpoint-year basket' $\{q_{t/2}\}$ is taken as follows (See Annex 1).

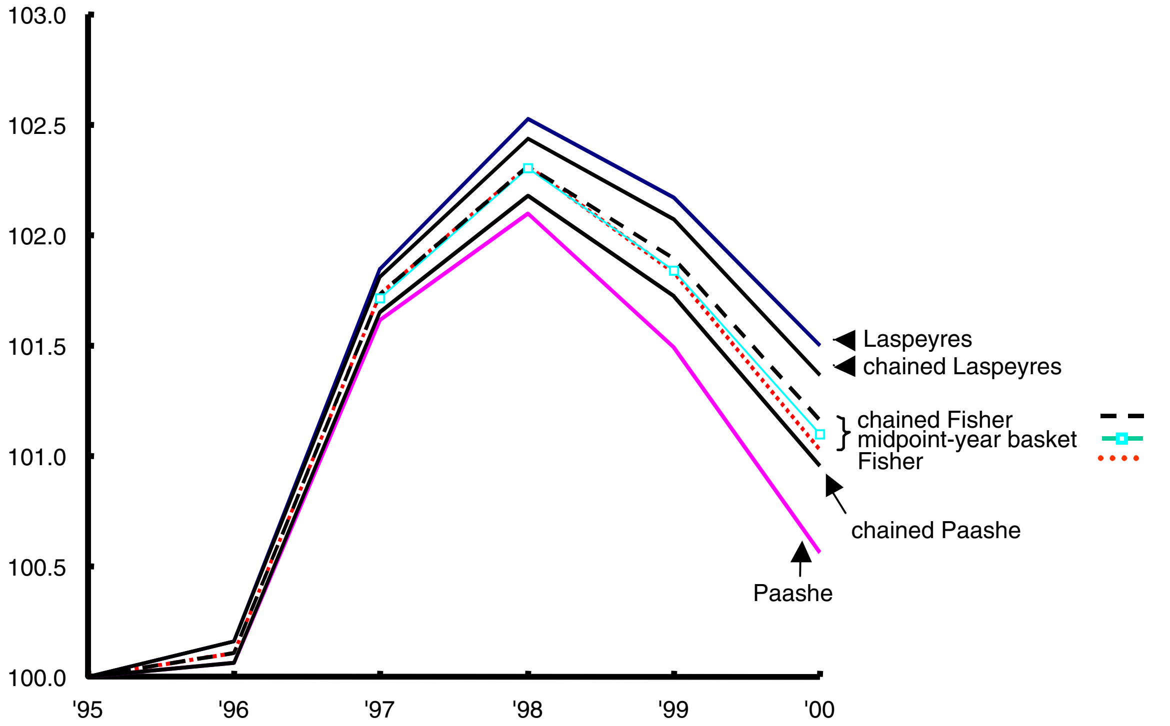
- The observation year 1997, 1999 ('single year' cases)
  The quantities in 1996, 1997 are used as contents in a basket respectively.

- The observation year 1996, 1998 ('plural year' cases)
  As the 'midpoint-year' is between two calendar years 1995 and 1996, 1996 and 1997 respectively, the simple geometric mean or the simple arithmetic mean of the quantities in the two respective years are used as contents in a basket respectively.

As shown in the following chart and table, the 'midpoint-year basket index' is very close to superlative indexes and chained superlative indexes in comparison with Laspeyres index or chained Laspeyres index. In 1998 or 2000 ('plural year' cases), the 'midpoint-year basket index' using the simple geometric mean and that of the simple arithmetic mean of the quantities in 1996 and 1997, or 1997 and 1998 are almost the same.

It seems to be possible to use an arithmetic mean or a geometric mean of the quantities in all intermediate years between the base year and the observation year as contents in a basket instead of the quantities in a single year. In fact, as shown in the table on the next page, both indexes are very close to each other although it is not clear that one is better than the other. Thus, the index using some average basket in all intermediate years may be possibly applicable to the case that weight data for individual year are not sufficiently accurate but average weights for two or three years are sufficiently accurate.

Comparison of consumer price changes measured by different index formulas



Results of the test calculation for consumer price index (the overall index)

	Laspeyres	Paasche	Fisher	Tornqvist	Walsh	Edgeworth	mid-year basket		average basket of all intermediate years	
'95	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000		
'96	100.162	100.061	100.111	100.112	100.112	100.111	100.112 [a]	100.111 [b]		
'97	101.844	101.618	101.731	101.738	101.739	101.731	101.715			
'98	102.523	102.095	102.309	102.318	102.322	102.311	102.308 [a]	102.306 [b]		
'99	102.169	101.492	101.830	101.857	101.862	101.835	101.844		101.855 [c]	101.853 [d]
'00	101.503	100.560	101.030	101.074	101.087	101.039	101.094	101.094	101.078 [c]	101.071 [d]

	chained Laspeyres	chained Paasche	chained Fisher	chained Tornqvist
'95	100.000	100.000	100.000	100.000
'96	100.162	100.061	100.111	100.112
'97	101.817	101.652	101.734	101.736
'98	102.441	102.175	102.308	102.308
'99	102.072	101.721	101.896	101.897
'00	101.362	100.957	101.159	101.160

a) Using (9) in Annex 1.
b) Using (10) in Annex 1.
c) Geometric mean of baskets in all intermediate years is used as a fixed basket.The relevant formula is similar to (9) in Annex1.
d) Arithmetic mean of baskets in all intermediate years is used as a fixed basket.The relevant formula is similar to (10) in Annex1.

The ‘midpoint-year basket index’ is considered to be a good price index for practical uses because of its features listed below, in addition that it yields very close approximate to superlative indexes and chained superlative indexes:

(i) Quantities required for the index compilation are in the ‘midpoint-year’ earlier than the observation year.[1)] Thus, the index compilation is feasible in countries where chained Laspeyres index is available.

(ii) The overall index can be expressed as a weighted arithmetic mean of sub-indexes. Thus, contribution of sub-item-groups to change in the overall index is available.

$$H = \frac{1}{\sum_{\text{all } i} \frac{w_{t/2,i}}{I_{t/2,i}}} \sum_{G} \left(\sum_{i \in G} \frac{w_{t/2,i}}{I_{t/2,i}} \cdot \frac{\sum_{i \in G} \frac{w_{t/2,i}}{I_{t/2,i}} I_{t,i}}{\sum_{i \in G} \frac{w_{t/2,i}}{I_{t/2,i}}} \right) = \frac{\sum_{G} w_G H_G}{\sum_{G} w_G} \qquad (2)$$

where $w_G = \sum_{i \in G} \frac{w_{t/2,i}}{I_{t/2,i}}$ ...... a weight assigned to sub - group G

$$H_G = \frac{\sum_{i \in G} \frac{w_{t/2,i}}{I_{t/2,i}} I_{t,i}}{\sum_{i \in G} \frac{w_{t/2,i}}{I_{t/2,i}}}$$ ...... 'midpoint - year basket index' for sub - group G

(iii) Monthly index can be defined so as the annual simple arithmetic mean is equal to the (annual) ‘midpoint-year basket index’ as shown below.

$$H = \frac{\sum \frac{w_{t/2}}{I_{t/2}} I_{\text{year } t}}{\sum \frac{w_{t/2}}{I_{t/2}}} = \frac{1}{12} \left(\sum_{\substack{\text{month } m \\ \text{of year } t}} \frac{\sum \frac{w_{t/2}}{I_{t/2}} I_{\text{month } m \text{ of year } t}}{\sum \frac{w_{t/2}}{I_{t/2}}} \right) \qquad (3)$$

where $I_{\text{year } t}$ is the annual price index in year t,

$I_{\text{month } m \text{ of year } t}$ is the monthly price index in month m of year t

$$I_{\text{year } t} = \frac{1}{12} \sum_{\substack{\text{month } m \\ \text{of year } t}} I_{\text{month } m \text{ of year } t}$$

(iv) The ‘midpoint-year basket index’ can be interpreted as the product of Laspeyres index with the base year *t/2* and the observation year *t*, and Paashe index with the base year 0 and the observation year *t/2*. Obviously this property stands up at sub-index level

[1)] In the year after the base year, the ‘midpoint-year basket index’ cannot be calculated timely. Thus, the index compilation procedures possibly have to be changed in some way if the ‘midpoint-year basket index’ is adopted. For example, the base revision is to be carried out two years after the base year or later.

(See (ii)). For this reason, addition of basic components can be carried out in intermediate years between the periodical base revision easily and more effectively in comparison with base-fixed basket indexes because weights for the Laspeyres index with the base year *t/2* can be revised while weights for the Paashe index with the observation year *t/2* remain un-revised.

$$H = \frac{\sum \frac{w_{t/2}}{I_{t/2}} I_t}{\sum \frac{w_{t/2}}{I_{t/2}}} = \frac{\sum w_{t/2}\left(\frac{p_t}{p_{t/2}}\right)}{\sum w_{t/2}} \frac{\sum w_{t/2}}{\sum \frac{w_{t/2}}{\left(\frac{p_{t/2}}{p_0}\right)}} \tag{4}$$

Supposing prices are observed on continuous time basis, and prices and quantities change smoothly, it can be proved by purely mathematical operations that the following CES type indexes which uses quantities at the midpoint-period *t/2* are second order differential approximations to superlative indexes and Divisia index at the base period 0 with respect to time (See Annex 2).[2)] In case of σ=0, this CES type index can be regarded as the 'midpoint-period basket index'. The 'midpoint-year basket index' is a kind of the 'midpoint-period basket index' on a discrete time basis. Thus, the 'midpoint-year basket index' is probably a good price index from a theoretical viewpoint also.

$$\text{'midpoint - period method' CES type index}$$

$$H = \left(\frac{\sum \frac{w_{t/2}}{I_{t/2}^{1-\sigma}} I_t^{1-\sigma}}{\sum \frac{w_{t/2}}{I_{t/2}^{1-\sigma}}} \right)^{\frac{1}{1-\sigma}} \tag{5}$$

$$\text{where } \sigma \neq 1$$

Apart from practical uses, it is a matter of interest which type of index formula incorporated with the 'midpoint-year method' is the best from a theoretical viewpoint. This question seems to be difficult to answer. However, there might be some relation between price elasticity of demand and choices of index formulas. Supposing $w_t = w_0 I_t^{1-\sigma}$ - i.e. constant elasticity, (5) is identical to the following base-fixed CES type index, chained CES type index, a superlative index defined as (11) in Annex 2 and its chained-index version. It is also identical to Divisia index if prices and quantities are observed on continuous time basis. Thus, if choosing an appropriate σ, (5) may be optimal.

2) We need to somewhat forget about the reality such as the existence of seasonal changes in this argument.

base - fixed CES type index

$$M = \left(\frac{\sum w_0 I_t^{1-\sigma}}{\sum w_0} \right)^{\frac{1}{1-\sigma}} \tag{6}$$

where $\sigma \neq 1$

Although the assumption given on the preceding paragraph looks unrealistic, according to a test calculation, all three indexes - base-fixed, chained and 'midpoint-year method' CES type indexes - can be regarded as good approximations to chained superlative indexes seemingly if choosing an appropriate σ around 0.75 or higher, where difference between base-fixed and chained CES type index is about the smallest, as shown in the chart on the next page. Furthermore, this choice of σ results in that 'midpoint-year method' CES type index becomes a slightly better approximation to chained superlative indexes in comparison with the 'midpoint-year basket index' (σ=0). It may be possible to find more relevant and complicated index formula such as (18) in Annex 2, seeking an appropriate estimate of price elasticity for each subgroup. However, as any σ between 0 through 1 gives a sufficiently accurate approximation, the necessity of search for appropriate price elasticity parameters for practical uses instead of the 'midpoint-year basket index' (σ=0) is likely to be weak. One notable fact obtained from the test calculation is that the 'midpoint-year method' yields better approximations to chained superlative indexes in comparison with superlative indexes in the observation year 1999 and 2000 – periods relatively far from the base year 1995. It may be attributable to a feature of the 'midpoint-year method'. That is, the 'midpoint-year method' index can be regarded as a kind of chained index consists of two indexes linked at the 'midpoint-year'.

Several types of combination of the 'midpoint-year method' with chain index method may be conceivable. A test calculation shows the following chained 'midpoint-year basket index' - the product of the 'midpoint-year basket index' with the base year 1995 and the observation year 1997, and that of the base year 1997 and the observation year 1999 - yields a slightly closer approximation to chained superlative indexes.

$$\text{chained } H(1999:1995) = \frac{\sum \frac{w_{1996}}{I_{1996}} I_{1997}}{\sum \frac{w_{1996}}{I_{1996}}} \cdot \frac{\sum \frac{\frac{w_{1998}}{I_{1998}} \frac{I_{1999}}{I_{1997}}}{I_{1997}}}{\sum \frac{\frac{w_{1998}}{I_{1998}}}{I_{1997}}} = 101.870 \tag{7}$$

where the base year is 1995, the observation year is 1999.

In the 2000 Japanese CPI revision, the 'midpoint-year basket index' will be added to a set of supplementary indexes, which includes chained Laspeyres index and indexes for the specific household groups, and it will be compiled annually. As for 'plural year' cases explained on page 2, formula (10) shown in Annex 1, which uses the simple arithmetic mean of quantities in two respective years as the 'midpoint-year basket', will be adopted taking it consideration that the possibility of monthly compilation in the future and the treatment for seasonally variable weights used for categories of fresh foods.

Comparison of indexes using 'midpoint-year method' with CES and superlative indexes

	$\sigma=0$					$\sigma=0.5$				
	Laspeyres	chained Laspeyres	midpoint-year [a]	Fisher	chained Fisher	base-fixed	chained	midpoint-year [a]	superlative [c]	chained [d] superlative
'95	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000
'96	100.162	100.162	100.112	100.111	100.111	100.134	100.134	100.112	100.112	100.112
'97	101.844	101.817	101.715	101.731	101.734	101.775	101.766	101.723	101.736	101.736
'98	102.523	102.441	102.308	102.309	102.308	102.371	102.341	102.290	102.316	102.308
'99	102.169	102.072	101.844	101.830	101.896	102.003	101.948	101.873	101.848	101.896
'00	101.503	101.362	101.094	101.030	101.159	101.282	101.210	101.142	101.062	101.160

	$\sigma=0.7$					$\sigma=0.75$				
	base-fixed	chained	midpoint-year [a]	superlative [c]	chained [d] superlative	base-fixed	chained	midpoint-year [a]	superlative [c]	chained [d] superlative
'95	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000
'96	100.122	100.122	100.112	100.112	100.112	100.119	100.119	100.112	100.112	100.112
'97	101.747	101.746	101.725	101.737	101.736	101.740	101.741	101.726	101.737	101.736
'98	102.310	102.302	102.283	102.317	102.308	102.294	102.292	102.281	102.317	102.308
'99	101.935	101.899	101.883	101.853	101.897	101.917	101.886	101.886	101.854	101.897
'00	101.189	101.149	101.159	101.069	101.161	101.165	101.134	101.163	101.071	101.161

	$\sigma=0.9$					$\sigma=1$				
	base-fixed	chained	midpoint-year [a]	superlative [c]	chained [d] superlative	geometric-mean	chained geometric-mean	midpoint-year [b]	Tornqvist	chained Tornqvist
'95	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000
'96	100.111	100.111	100.112	100.112	100.112	100.105	100.105	100.112	100.112	100.112
'97	101.718	101.725	101.728	101.738	101.736	101.703	101.715	101.729	101.738	101.736
'98	102.248	102.262	102.276	102.318	102.308	102.217	102.242	102.272	102.318	102.308
'99	101.865	101.849	101.894	101.856	101.897	101.830	101.825	101.898	101.857	101.897
'00	101.094	101.088	101.175	101.074	101.161	101.045	101.058	101.181	101.074	101.160

Difference from chained Tornqvist

	$\sigma=0$					$\sigma=0.5$				
	Laspeyres	chained Laspeyres	midpoint-year [a]	Fisher	chained Fisher	base-fixed	chained	midpoint-year [a]	superlative [c]	chained [d] superlative
'97	0.108	0.081	-0.021	-0.005	-0.002	0.039	0.031	-0.013	0.000	0.000
'98	0.215	0.133	0.000	0.001	0.000	0.063	0.033	-0.018	0.007	0.000
'99	0.272	0.175	-0.053	-0.067	0.000	0.106	0.051	-0.024	-0.048	0.000
'00	0.342	0.202	-0.066	-0.130	-0.001	0.121	0.050	-0.019	-0.098	0.000

	$\sigma=0.7$					$\sigma=0.75$				
	base-fixed	chained	midpoint-year [a]	superlative [c]	chained [d] superlative	base-fixed	chained	midpoint-year [a]	superlative [c]	chained [d] superlative
'97	0.011	0.010	-0.011	0.001	0.000	0.004	0.005	-0.010	0.001	0.000
'98	0.002	-0.007	-0.025	0.009	0.000	-0.014	-0.016	-0.027	0.009	0.000
'99	0.038	0.002	-0.013	-0.044	0.000	0.021	-0.011	-0.011	-0.043	0.000
'00	0.028	-0.011	-0.001	-0.091	0.000	0.005	-0.026	0.003	-0.090	0.000

	$\sigma=0.9$					$\sigma=1$				
	base-fixed	chained	midpoint-year [a]	superlative [c]	chained [d] superlative	geometric-mean	chained geometric-mean	midpoint-year [b]	Tornqvist	chained Tornqvist
'97	-0.018	-0.010	-0.008	0.002	0.000	-0.033	-0.021	-0.006	0.002	0.000
'98	-0.060	-0.046	-0.032	0.010	0.000	-0.091	-0.066	-0.036	0.010	0.000
'99	-0.031	-0.048	-0.003	-0.040	0.000	-0.067	-0.072	0.002	-0.039	0.000
'00	-0.067	-0.072	0.015	-0.087	0.000	-0.115	-0.103	0.021	-0.086	0.000

a) As for 1996, 1998, geometric-mean of weights and elementary indexes for 1995 and 1996, 1996 and 1997 are used for 'midpoint-year' weights and indexes respectively.
b) As for 1996, 1998, arithmetic-mean of weights for 1995 and 1996, 1996 and 1997 are used for 'midpoint-year' weights respectively.
c) Index formula defined as (11) in Annex 2 is used.
d) Chained-index version of (11)

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Annex 1. Formula of the midpoint-year basket index used for the test calculation (the base year : 1995)

In the case of the year 1997, 1999:

$$H=\frac{\sum q_m p_t}{\sum q_m p_0}=\frac{\sum p_m q_m \frac{p_t/p_0}{p_m/p_0}}{\sum p_m q_m \frac{1}{p_m/p_0}}=\frac{\sum \frac{w_m}{I_m} I_t}{\sum \frac{w_m}{I_m}} \tag{8}$$

where m is 1996, 1997 respectively.

In the case of the year 1996, 1998:

$$H=\frac{\sum \sqrt{q_m q_{m+1}}\, p_t}{\sum \sqrt{q_m q_{m+1}}\, p_0}=\frac{\sum \sqrt{p_m q_m p_{m+1} q_{m+1}}\ \frac{\frac{p_t}{p_0}}{\sqrt{\frac{p_m}{p_0}\frac{p_{m+1}}{p_0}}}}{\sum \sqrt{p_m q_m p_{m+1} q_{m+1}}\ \frac{1}{\sqrt{\frac{p_m}{p_0}\frac{p_{m+1}}{p_0}}}}=\frac{\sum \sqrt{\frac{w_m}{I_m}\frac{w_{m+1}}{I_{m+1}}}\cdot I_t}{\sum \sqrt{\frac{w_m}{I_m}\frac{w_{m+1}}{I_{m+1}}}} \tag{9}$$

where $m,\ m+1$ are 1995 and 1996, 1996 and 1997 respectively

or

$$H=\frac{\sum (q_m+q_{m+1})p_t}{\sum (q_m+q_{m+1})p_0}=\frac{\sum p_m q_m \frac{p_t/p_0}{p_m/p_0}+p_{m+1}q_{m+1}\frac{p_t/p_0}{p_{m+1}/p_0}}{\sum p_m q_m \frac{1}{p_m/p_0}+p_{m+1}q_{m+1}\frac{1}{p_{m+1}/p_0}}=\frac{\sum \left(\frac{w_m}{I_m}+\frac{w_{m+1}}{I_{m+1}}\right) I_t}{\sum \frac{w_m}{I_m}+\frac{w_{m+1}}{I_{m+1}}} \tag{10}$$

where $m,\ m+1$ are 1995 and 1996, 1996 and 1997 respectively.

Annex 2. Relation between superlative indexes and midpoint-period basket index

We assume prices and quantities consumed by households change smoothly. Define the following superlative price indexes $S(t{:}0)$, and CES type price indexes $H(t{:}0)$ which uses prices and quantities at the midpoint-period $t/2$:

$$S(t:0)=\left(\frac{\sum s_0 I_t^{\frac{r}{2}}}{\sum \frac{s_t}{I_t^{\frac{r}{2}}}}\right)^{\frac{1}{r}}=\left(\frac{\sum s_0 I_t^{1-\sigma}}{\sum \frac{s_t}{I_t^{1-\sigma}}}\right)^{\frac{1}{2(1-\sigma)}} \tag{11}$$

or superlative index derived from any (flexible) linear homogeneous unit cost function $c(p)$ as follows [3]

$$S(t:0)=\frac{c(p_t)}{c(p_0)}=\frac{c(p_0 I_t)}{c(p_0)},\quad \frac{1}{\frac{c(p_0 I_t)}{c(p_0)}}\frac{\partial \frac{c(p_0 I_t)}{c(p_0)}}{\partial I_t}=\frac{s_t}{I_t} \tag{12}$$

or the following index derived from any positive second-order-differentiable function f, g satisfied with $f(1)=g(1)=1$

$$H(t:0)=\left(\frac{\sum \frac{s_{t/2}}{I_{t/2}^{\frac{r}{2}}} I_t^{\frac{r}{2}}}{\sum \frac{s_{t/2}}{I_{t/2}^{\frac{r}{2}}}}\right)^{\frac{2}{r}}=\left(\frac{\sum \frac{s_{t/2}}{I_{t/2}^{1-\sigma}} I_t^{1-\sigma}}{\sum \frac{s_{t/2}}{I_{t/2}^{1-\sigma}}}\right)^{\frac{1}{1-\sigma}} \tag{13}$$

$$H(t:0)=\frac{L(t:t/2)}{L(0:t/2)}=\frac{f^{-1}\left(\sum s_{t/2} f\left(\frac{I_t}{I_{t/2}}\right)\right)^{\alpha}}{f^{-1}\left(\sum s_{t/2} f\left(\frac{1}{I_{t/2}}\right)\right)^{\alpha}}\cdot\frac{g^{-1}\left(\sum s_t g\left(\frac{I_t}{I_{t/2}}\right)\right)^{1-\alpha}}{g^{-1}\left(\sum s_0 g\left(\frac{1}{I_{t/2}}\right)\right)^{1-\alpha}} \tag{14}$$

where $r=2(1-\sigma)\neq 0,\ I_{\bullet}=\frac{p_{\bullet}}{p_0},\ s_{\bullet}=\frac{p_{\bullet}q_{\bullet}}{\sum p_{\bullet}q_{\bullet}}$.

$$L(t:0)=f^{-1}\left(\sum s_0 f(I_t)\right)^{\alpha}\cdot g^{-1}\left(\sum s_t g(I_t)\right)^{1-\alpha}$$

[3] See Diewert's paper such as Diewert(1981).

(13) can be regarded as the 'midpoint-period basket index' in the case: r=2 or σ=0. $L(t{:}0)$ equals to (11) in the case: $f(x)=x^{r/2}$, $g(x)=1/x^{r/2}$ and α=1/2. We can obtain the following results by differentiating (11), (12), (13) and (14) with respect to time t at the base period 0.

$$\left.\frac{dS}{dt}\right|_{t=0} = \left.\frac{dH}{dt}\right|_{t=0} = \sum s_0 \left.\frac{dI_t}{dt}\right|_{t=0}$$

$$\left.\frac{d^2S}{dt^2}\right|_{t=0} = \left.\frac{d^2H}{dt^2}\right|_{t=0}$$

$$= \left(\sum s_0 \left.\frac{dI_t}{dt}\right|_{t=0}\right)^2 + \sum s_0 \left.\frac{d^2 I_t}{dt^2}\right|_{t=0} + \sum \left.\frac{ds_t}{dt}\right|_{t=0} \left.\frac{dI_t}{dt}\right|_{t=0} - \sum s_0 \left(\left.\frac{dI_t}{dt}\right|_{t=0}\right)^2 \tag{15}$$

or

$$\left.\frac{d^2 \ln S}{dt^2}\right|_{t=0} = \left.\frac{d^2 \ln H}{dt^2}\right|_{t=0} = \left.\frac{d}{dt}\left(\sum s_t \left(\frac{d \ln I_t}{dt}\right)\right)\right|_{t=0}$$

16)

using $\left.\frac{dI_{t/2}}{dt}\right|_{t=0} = \frac{1}{2}\left.\frac{dI_t}{dt}\right|_{t=0}$, $\left.\frac{ds_{t/2}}{dt}\right|_{t=0} = \frac{1}{2}\left.\frac{ds_t}{dt}\right|_{t=0}$

$$\left.\frac{d^2 I_{t/2}}{dt^2}\right|_{t=0} = \frac{1}{4}\left.\frac{dI_t}{dt}\right|_{t=0}, \quad \left.\frac{d^2 s_{t/2}}{dt^2}\right|_{t=0} = \frac{1}{4}\left.\frac{ds_t}{dt}\right|_{t=0}$$

$$\sum s_t = 1,\ \sum \frac{ds_t}{dt} = 0,\ \sum \frac{d^2 s_t}{dt^2} = 0,\ I_0 = 1$$

Obviously (13) and (14) are also second order differential approximations to Divisia index defined as follows at the base period 0. Index $H(t{:}0)$ can be described as 'flexible' because it satisfies the equation on the right side in (15) and (16) without imposing any condition with respect to relations between prices { p_t } and quantities { q_t }. Prices { p_t } dose not need to be related with quantities { q_t } by a concave unit cost function or a concave aggregator function.

$$Divisia(t:0) = \exp\left(\int_0^t \sum \frac{s_t}{I_t}\frac{dI_t}{dt}dt\right)$$

(13) converges on the following geometric-mean type index which uses weights at the midpoint-period $t/2$ when r approaches 0 or σ approaches 1. This index is also a second order differential approximation to superlative indexes with respect to time at the base period 0.

$$H(t:0) = \prod I_t^{\,s_{t/2}} \tag{17}$$

(13) and (14) can be further generalized as follows.

$$H(t:0)=\left(\frac{\sum_k a_{t/2,k}\left(\sum \frac{s_{t/2}}{I_{t/2}^{1-\sigma_k}} I_t^{1-\sigma_k}\right)^{\frac{1-\delta}{1-\sigma_k}}}{\sum_k a_{t/2,k}\left(\sum \frac{s_{t/2}}{I_{t/2}^{1-\sigma_k}}\right)^{\frac{1-\delta}{1-\sigma_k}}}\right)^{\frac{1}{1-\delta}} \tag{18}$$

or

$$H(t:0)=\frac{f^{-1}\left(\sum_k a_{t/2,k} f\left(\phi_k^{-1}\left(\sum s_{t/2}\phi_k\left(\frac{I_t}{I_{t/2}}\right)\right)\right)\right)^{\alpha}}{f^{-1}\left(\sum_k a_{t/2,k} f\left(\phi_k^{-1}\left(\sum s_{t/2}\phi_k\left(\frac{1}{I_{t/2}}\right)\right)\right)\right)^{\alpha}}\cdot\frac{g^{-1}\left(\sum_k a_{t,k}\, g\left(\varphi_k^{-1}\left(\sum s_t\varphi_k\left(\frac{I_t}{I_{t/2}}\right)\right)\right)\right)^{1-\alpha}}{g^{-1}\left(\sum_k a_{0,k}\, g\left(\varphi_k^{-1}\left(\sum s_0\varphi_k\left(\frac{1}{I_{t/2}}\right)\right)\right)\right)^{1-\alpha}} \tag{19}$$

where σ_k : price elasticity of demand for basic components in subgroup $k(\neq 0)$,

δ : price elasticity of demand for subgroups$(\neq 0)$

$s_\bullet$: share of basic component in subgroup k at period $\bullet$

$a_{\bullet,k}$: share of subgroup k in the total consumption at period $\bullet$

f , g, ϕ_k , φ_k : any positive second - order - differentiable function satisfied with $f(1)=g(1)=\phi_k(1)=\varphi_k(1)=1$

As well as (13) and (14), (18) and (19) are also second order differential approximations to superlative indexes with respect to time at the base period supposing prices and quantities change smoothly – i.e. both satisfy with (15) and (16).